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Fixed Point Results for Cyclic Contractive Mappings and Application to Volterra Integral Equations



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Abstract

Inspired by the fact that the discontinuous mappings cannot be (Banach type) contractions and cyclic contractions need not be continuous, and taking into account that there are applications to integral and differential equations based on cyclic contractions, we present results concerning the existence of fixed points for newly defined cyclic contractive mapping of an implicit relation in a metric space and derive existence and uniqueness results of fixed points for such mappings. Examples are given to support the usability of our results. Application to the study of existence and uniqueness of solutions for a class of nonlinear Volterra integral equations in two variables is presented.

Keywords: Fixed point, cyclic contraction, implicit relation, integral equation.
Subject Classification: Primary 54H25, Secondary 47H10.

1. Introduction and Preliminaries:

It is well known that the contraction mapping principle, formulated and proved in the Ph. D. dissertation of Banach in 1920, which was published in 1922 [1], is one of the most important theorems in classical functional analysis. The Banach Contraction Principle is a very popular tool which is used to solve existence problems in many branches of Mathematical Analysis and its applications. It is no surprise that there is a great number of generalizations of this fundamental theorem. They go in several directions modifying the basic contractive

condition or changing the ambiental space. This celebrated theorem can be stated as follows.
Theorem 1. [1]. Let (x, d) be a complete metric space and T be a mapping of X into itself satisfying:

(1.1) $d(Tx,Ty) \leq kd(x,y) \forall\, x,y \in X$,

where k is a constant in (0, 1) .Then, T has a unique fixed point $x^* \in X$.
There is in the literature a great number of generalizations of the Banach contraction principle (see, e.g., [2] and references cited therein).
Inequality (1.1) implies continuity of T. A natural question is whether we can find contractive conditionswhich will imply existence of a fixed point in a complete metric space but will not imply continuity.
On the other hand, cyclic representations and cyclic contractions were introduced by Kirk et al. [3].

Definition1. (see [3,8]) Let (x, d) be a metric space. Let p be a positive integer, and $A_1, A_2, \ldots \ldots . A_p$ be nonempty subsets of X . Then $Y = \cup_{i=1}^{p} A_i$ is said to be a cyclic representation of Y with respect to $T: Y \to Y$ if

(i). $A_i, i = 1,2, \ldots ., p$, are nonempty closed sets, and

(ii). $T(A_1) \subseteq A_2, \ldots, T(A_{p-1}) \subseteq A_p, T(A_p) \subseteq A_1$

Kirk et.al., [3] proved the following result.

Theorem2. ([3])Let (x, d) be a metric space and let $Y = \cup_{i=1}^{p} A_i$ be a cyclic representation of Y with respect to $T: Y \to Y$.If

$$d(Tx,Ty) \leq kd(x,y)$$

holds for all $x,y \in A_i \times A_{i+1}$, i =1 , 2 ,...,p (where $A_{p+1} = A_1$), and $0 \leq k < 1$, then T has a unique fixed point X^* and $X^* \in \cap_{i=1}^{p} A_i$,
Notice that although a contraction is continuous, cyclic contractions need not be. This is one of the important gains of this approach.
Following [3], a number of fixed point theorems on cyclic representations of Y with respect to a self-mapping T have appeared (see, e.g., [4] – [16]).
In this paper, we introduce a new class of cyclic contractive mappings satisfying an implicit relation in the framework of metric spaces, and then derive the existence and uniqueness of fixed points for such mappings. Suitable examples are provided to demonstrate the validity of our results. Our main result generalizes and improves many existing theorems in the literature. We also give an application of the presented results in the area of integral equations and prove an existence theorem for solutions of a nonlinear Volterra integral equations in two variables in the last section.

2. Notation And Definitions

First, we introduce some further notation and definitions that will be used later.

Implicit relation and related concepts. In recent years, Popa [17] used implicit functions rather than contraction conditions to prove fixed point theorems in metric spaces whose strength lies in its unifying power. Namely, an implicit function can cover several contraction conditions which include known as well as some new conditions. This fact is evident from examples furnished in Popa [17]. Implicit relations on metric spaces have been used in many articles (for details see [18, 19, 20, 22, 23, 24] and references cited therein).

In this section, we define a suitable implicit function involving six real non-negative arguments to prove our results, that was given in [21].

Let $\mathbb{R}_+$ denote the non-negative real numbers and let Φ be the set of all continuous functions $T(t_1, t_2, t_3, t_4, t_5, t_6): \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

($\emptyset_1$): $T(t_1, t_2, t_3, t_4, t_5, t_6)$: is non-increasing in variable t_5,

($\emptyset_h$): there exists $h \in [0,1)$ such that for $u, v \geq 0$,

$$T(u, v, v, u, u, 0) \leq 0$$

implies $u \leq hv$,

($\emptyset_u$): $T(u, u, 0, 0, u, u) > 0, \forall\, u > 0$.

Example1. $T(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - at_5t_6 - b\max\{t_2^2, t_3^2, t_4^2)$

where $a > 0, b \geq 0$ and $a + b < 1$, or

Example2. $T(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - at_5t_6 - t_1(bt_2 + ct_3 + dt_4)$

where $a, b, c, d \geq 0$ and $a + b + c + d < 1$.

Example3. $T(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^3 - at_1^2t_2 - bt_2^2t_1 - ct_2t_3t_4 - dt_5^2t_6)$

where $a > 0, b, c, d \geq 0$ and $a + b + c + d < 1$

Now we introduce a new notion of cyclic contractive mapping satisfying an implicit relation.

Definition 2. Let (x, d) be a metric space. Let p be a positive integer, $A_1, A_2, \ldots \ldots . A_p$ be nonempty subsetsof X and $Y = \cup_{i=1}^{p} A_i$. An operator $F: Y \to Y$ is called cyclic contractive mapping of an implicit relation type (in short, CCIRT) if

($*$) $Y = \cup_{i=1}^{p} A_i$ is a cyclic representation of Y with respect to F ,

($**$) for any $x, y \in A_i \times A_{i+1}$, i =1 , 2 ,...,p (where $A_{p+1} = A_1$),

(2.1) $T(d(Fx, Fy), d(x,y), d(x, Fx), d(y, Fy), d(y, F^2x), d(y, Fx)) \leq 0$,

for some $T \in \Phi$

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Using Example 1, we present an example of cyclic contractive mapping of an implicit relation type.

Example 4. Let $X=[0,1]$ with the usual metric. Suppose $A_1=[0,\frac{3}{4}]$, $A_2=[\frac{3}{4},1]$, note that $X=\bigcup_{i=1}^{p}A_i$.
Define $F:X\to X$ such that

(2.2) $$Fx=\begin{cases}\frac{3}{4}, & x\in[0,1)\\ 0 & x=1\end{cases}$$

Clearly, A_1 and A_2 are closed subsets of X . Moreover $F(A_i)\subset A_{i+1}$ for i = 1 , 2, so that $\bigcup_{i=1}^{2}A_i$ is a cyclic representation of X with respect to F . Furthermore, if $T:\mathbb{R}^{+6}\to\mathbb{R}^{+}$is given by

(2.3) $$T(t_1,t_2,t_3,t_4,t_5,t_6)=t_1^2-at_5t_6-b\max\{t_2^2,t_3^2,t_4^2)$$

then $T\in\Phi$. It is clear that F is a CCIRT for a $=\frac{1}{3}$, b = $\frac{1}{2}$.

3. Main Result

Our main result is the following.

Theorem3. Let (x,d) be a complete metric space, $p\in\mathbb{N}$, $A_1,A_2,\ldots\ldots.A_p$ nonempty closed subsets of X and $Y=\bigcup_{i=1}^{p}A_i$. Suppose $F:Y\to Y$ is CCIRT type mapping, for some $T\in\Phi$. Then F has a unique fixed point. Moreover, the fixed point of F belongs to $\bigcap_{1=1}^{p}A_i$.

Proof. Let $x_0\in A_1$ (such a point exists since $A_1\neq\emptyset$). Define the sequence $\{x_n\}$ in X by:

$$x_{n+1}=Fx_n,\ \ n=0,1,2,\cdots.$$

We shall prove that

(3.1) $$\lim_{n\to\infty}d(x_n,x_{n+1})=0.$$

If for some k, we have $x_{k+1}=x_k$, then (3.1) follows immediately. So, we can suppose that $d(x_n,x_{n+1})>0$ for all n. From the condition ($*$) , we observe that for all n, there exists i = i(n) $\in\{1,2,\cdots,p\}$ such that $(x_n,x_{n+1})\in A_i\times A_{i+1}$. Then, from the condition ($**$),we have

$$T(d(Fx_n,Fx_{n+1}),d(x_n,x_{n+1}),d(x_n,Fx_n),d(x_{n+1},Fx_{n+1}),d(x_{n+1},Fx_n),d(x_{n+1},Fx_n))\le 0$$

and so

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$T(d(Fx_n, F^2x_n), d(Fx_n, Fx_n), d(Fx_n, Fx_n), d(Fx_n, F^2x_n), d(Fx_n, F^2x_n), 0) \leq 0.$
Now since T ∈ Φ and we have

$$T(u, v, v, u, u, 0) \leq 0$$

for $= d(Fx_n, F^2x_n)$, $v = d(x_n, Fx_n)$, it follows from ($\emptyset_h$) that there exists h ∈ [0, 1)such that

$$d(Fx_n, F^2x_n) \leq hd(x_n, Fx_n).$$

By $x_{n+1} = \mathrm{F}^{\mathrm{n}}\mathrm{x}$, n = 0,1,2, … …, we have

$$d(x_{n+1}, x_{n+2}) \leq hd(x_n, x_{n+1}).$$

If we take $(x_{n+1}, x_n) \in A_{i+1} \times A_i$ then

$$d(x_{n+2}, x_{n+1}) \leq hd(x_{n+1}, x_n).$$

On combining, we have

$$d(x_{n+1}, x_n) \leq hd(x_n, x_{n-1}).$$

If we continue this procedure, we can have

(3.2) $$d(x_{n+1}, x_n) \leq h^n(d(x_1, x_0)).$$

The sequence $a_n := d(x_n, x_{n+1}), n \geq 1$, of non-negative real numbers, converges to zero. Then there exists a bijection $\sigma: N^* \to N$ such that the sequence $\{a_{\sigma(n)}\}$ is non-increasing. It follows that the sequence $\{d(x_{\sigma(n)}, x_{\sigma(n)+1})\}$of non - negative real numbers is non-increasing, hence it is convergent. It follows that $\lim_{n\to\infty} d(x_{\sigma(n)}, x_{\sigma(n)+1}) = 0$, therefore

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

Next we show that $\{x_n\}$ is a Cauchy sequence. Suppose it is not true. Then we can find $a \in_0 > 0 >$ and two sequence of integers $\{m_k\}$,$\{n_k\}$,$m_k \geq n_k \geq k$ with

(3.3) $$r_k = d(x_{m_k}, x_{n_k}) \geq \in_0 \ for k \in \{1,2,3, \dots \dots\}.$$

We may also assume

(3.4) $$d(x_{m_k - 1}, x_{n_k}) < \in_0$$

by choosing m_k to be the smallest number exceeding n_k for which (3.3) holds. Now (3.2), (3.3) and (3.4) imply

$$\begin{aligned} \in_0 \leq r_k &\leq d(x_{m_k}, x_{m_k - 1}) + d(x_{m_k - 1}, x_{n_k}) \\ &\leq h^{m_k - 1}(d(x_1, x_0)) + \in_0 \end{aligned}$$

and so

(3.5) $$\lim_{k\to\infty} r_k = \in_0.$$

Also since

$$\in_0 \leq r_k \leq d(x_{n_k}, x_{n_k+1}) + d(x_{m_k}, x_{m_{k+1}}) + d(x_{n_{k+1}}, x_{m_k+1})$$

we have from (3.2) that

(3.6) $$\in_0 \leq r_k \leq h^{n_k}(d(x_1, x_0)) + h^{m_k}(d(x_1, x_0)) + d(x_{n_{k+1}}, x_{m_k+1}).$$

On the other hand, for each $k \in N$, we may find $j_k \in \{1,2,3 \ldots, p\}$ in which $n_k - m_k + j_k \equiv 1 (mod p)$.
For k large enough, we may see that $m_k + j_k > 0$. Consider that

$$\begin{aligned} \left|d(x_{m_k - j_k}, x_{n_k}) - d(x_{n_k}, x_{m_k})\right| &\le d(x_{m_k - j_k}, x_{m_k}) \\ &\le \sum_{l=0}^{j_k-1} d(x_{m_k - j_k + l}, x_{m_k - j_k + l + 1}) \\ &\le \sum_{l=0}^{p-1} d(x_{m_k - j_k + l}, x_{m_k - j_k + l + 1}). \end{aligned}$$

Passing to the limit as $k \to \infty$, we consequently have

(3.7) $$d(x_{m_k - j_k}, x_{n_k}) \to \in_0$$

Also consider that

$$\left|d(x_{n_k}, x_{m_k - j_k}) - d(x_{m_k - j_k}, x_{n_k+1})\right| \le d(x_{n_k}, x_{n_k+1}).$$

As $k \to \infty$, we have

(3.8) $$d(x_{m_k - j_k}, x_{n_k+1}) \to \in_0.$$

Similarly, we have

$$\left|d(x_{m_k - j_k}, x_{n_k}) - d(x_{n_k}, x_{m_k - j_k + 1})\right| \le d(x_{m_k - j_k}, x_{m_k - j_k + 1}).$$

So, we have

(3.9) $$d(x_{n_k}, x_{m_k - j_k + 1}) \to \in_0$$

as $k \to \infty$. Also observe that

$$\left|d(x_{n_k}, x_{n_k+1}) - d(x_{n_k+1}, x_{m_k - j_k + 1})\right| \le d(x_{n_k}, x_{m_k - j_k + 1}).$$

Again, passing to the limit as $k \to \infty$, we obtain that

(3.10) $$d(x_{n_k+1}, x_{m_k - j_k + 1}) \to \in_0.$$

Finally, by the fact that $(x_{m_k - j_k}, x_{n_k}) \in A_i \times A_{i+1}$ for some $i \in \{1,2,3, \ldots. p\}$ and (2.1), we may obtain that

$T(d(Fx_{m_k - j_k}, Fx_{n_k}), d(x_{m_k - j_k}, x_{n_k}), d(x_{m_k - j_k}, Fx_{n_k}), d(x_{n_k}, Fx_{n_k}), d(x_{n_k},$
F2xmk−jk,dxnk,Fxmk−jk≤0

and so

$T(d(Fx_{m_k - j_k}, Fx_{n_k}), d(x_{m_k - j_k}, x_{n_k}), d(x_{m_k - j_k}, Fx_{n_k}), d(x_{n_k}, x_{n_k+1}), d(x_{n_k},$
Fxmk−jk+2,dxnk,xmk−jk+1≤0.

Now passing to the limit $k \to \infty$ and using (3.5)-(3.10) we have, by continuity of T, that

$$T(\epsilon_0, \epsilon_0, 0, 0, \epsilon_0, \epsilon_0) \le 0.$$

A contradiction with ($\emptyset_u$) since we have supposed that $\epsilon_0 > 0$. Thus $\{x_n\}$ is a Cauchy sequence in X. Since (x, d) is complete, there exists $x \in X$ such that

(3.11) $$\lim_{n\to\infty} x_n = x^*.$$

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We shall prove that

(3.12) $\qquad x^* \in \cap_{i=1}^{p} A_i$.

From condition ($*$), and since $x_0 \in A_1$, we have $\{x_{np}\}_{n\geq 0} \subseteq A_1$. Since A_1 is closed, from (3.11), we get that $x^* \in A_1$. Again, from the condition ($*$), we have $\{x_{np+1}\}_{n\geq 0} \subseteq A_2$. Since A_2 is closed, from (3.11), we get that $x^* \in A_2$. Continuing this process, we obtain (3.12).

Now, we shall prove that x^* is a fixed point of F . Indeed, from (3.12), for all n, there exists $i \in \{1,2,3,\ldots.p\}$ such that $x_n \in A_{i(n)}$. Applying ($**$) with $x = x_n$ and $y = x^*$ we obtain

$$T(d(Fx_n, Fx^*), d(x_n, x^*), d(x_n, Fx_n), d(x^*, x^*), d(x^*, F^2x_n), d(x^*, Fx_n) \leq 0$$

and so

$$T(d(x_{(n+1)}, Fx^*), d(x_n, x^*), d(x_n, x_{(n+1)}), d(x^*, Fx^*), d(x^*, x_{(n+2)}), d(x^*, x_{(n+1)})) \leq 0.$$

Passing to the limit as n $\rightarrow\infty$ from the last inequality, we also have

$$T(d(x^*, Fx^*), 0,0, d(x^*, Fx^*), 0,0) \leq 0,$$

$$T(d(x^*, Fx^*), 0,0, d(x^*, Fx^*), d(x^*, Fx^*), 0) \leq 0,$$

which is a contradiction to ($\emptyset_h$). Thus $d(x^*, Fx^*) = 0$ and so $x^* = Fx^*$, that is, $x^*$is a fixed point of F.

Finally, we prove that $x^*$is the unique fixed point of F. Assume that y is another fixed point of F,

that is, $y^* = Fy^*$. By the condition ($*$), this implies that $y^* \in \cap_{i=1}^{p} A_i$.Then we can apply ($**$) for $x = x^*$and $y = y^*$.Hence, we obtain

$$T(d(Fx^*, Fy^*), d(x^*, y^*), d(x^*, Fx^*), d(y^*, Fy^*), d(y^*, F^2x^*), d(y^*, Fx^*)) \leq 0$$

Since $x^*$and $y^*$are fixed points of F,we can show easily that $x^* \neq y^*$.If $d(x^*, y^*) > 0$, we get

$$T(d(x^*, y^*), d(x^*, y^*) 0,0, d(y^*, x^*), d(y^*, x^*)) \leq 0$$

which is a contradiction to ($\emptyset_u$). Then we have $d(x^*, y^*) = 0$, that is, $x^* = y^*$.Thus we have proved theuniqueness of the fixed point.

In the following, we deduce some fixed point theorems from our main result given by Theorem 3.

If we take $p = 1$ and $A_1 = X$ in Theorem 3, then we get immediately the following fixed point theorem.

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Corollary1. Let $(x,\ d)$ be a complete metric space and let $F: X \to \mathrm{X}$ satisfy the following condition: there exists $T \in \Phi$ such that

$$T\,(d(Fx, Fy), d(x, y), d(x, Fx), d(y, Fy), d(y, F^2x), d(y, Fx)) \le 0$$

for all $x, y \in X$. Then F has a unique fixed point.

Corollary2. Let $(x,\ d)$ be a complete metric space, $p \in \mathbb{N}$, $A_1, A_2, \ldots \ldots A_p$ nonempty closed subsets of X and $Y = \cup_{i=1}^{p} A_i$. and $F: Y \to \mathrm{Y}$. Suppose that there exists $T \in \Phi$. such that

$(*)$ $Y = \cap_{i=1}^{p} A_i$ is a cyclic representation of Y with respect to F ;

$(**)$ for any $x, y \in A_i \times A_{i+1}$, i =1 , 2 ,…,p (where $A_{p+1} = A_1$),

(3.13) $$d^2(Fx, Fy) \le a\, d(y, F^2x)\, d(y, Fx) + b \max\{d^2(x, y), d^2(x, Fx), d^2(y, Fy)\},$$

where $a > 0, b \ge 0$ and $a + b < 1$, or

(3.14)

$$d^2(Fx, Fy) \le a d(y, F^2x)\, d(y, Fx)\ \ d(x, Fx) + d(Fx, Fy)[bd(x, y) + cd(x, Fx) + d\,d(y, Fy)],$$

where $a, b, c, d \ge 0$ and $a + b + c + d < 1$, or

(3.15) $$\begin{aligned} &d^3(Fx, Fy) - ad^2(Fx, Fy)\, d(x, y) - bd(Fx, Fy)\, d^2(x, y) \\ &- cd(x, y)\, d(x, Fx)\, d(y, Fy) - d\, d^2(y, F2x)\, d(y, Fx) \le 0 \end{aligned}$$

where $a > 0, b, c, d \ge 0$ and $a + b + c + d < 1$.

Then F has a unique fixed point. Moreover, the fixed point of F belongs to $\cap_{i=1}^{p} A_i$.

The following examples demonstrate the validity of Theorem 3:

Example5. Let $X = R$ with the usual metric. Suppose $A_1 = [-2,0] = A_3$ and $A_2 = [0,2] = A_4$ and $Y = \cap_{i=1}^{4} A_i$. Define $F: Y \to \mathrm{Y}$ by $F_x = \frac{-x}{5}$ for all $x \in Y$. Clearly, $A_i (i = 1,2,3,4)$ are closed subsets of X. Moreover, mapping F is cyclic contractive mapping of an implicit relation type with $T: \mathbb{R}^{+^n} \to \mathbb{R}^+$defined by

$$T(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - at_5 t_6 - t_1(bt_2 + ct_3 + dt_4)$$

For $a = \frac{1}{3}$, $b = c = d = \frac{1}{6}$ such that $a + b + c + d < 1$.

Therefore, all conditions of Theorem 3are satisfied, and so F has a fixed point (which is $z = 0 \in \cap_{i=1}^{p} A_i$).

Example6. Let $X = \mathbb{R}^{+^3}$and we define $d: X \times X \to [0,1)$ by

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2 \| x_3 - y_3|, \text{ for } x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in X,$$

and let A = {(x, 0, 0) : x ∈ $\mathbb{R}^+$}, B = {(0, y, 0) : y ∈ $\mathbb{R}^+$}, C = {(0, 0, z) : z ∈ $\mathbb{R}^+$} be three subsets of X.

Define $F: A \cup B \cup C \to A \cup B \cup C$ by

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$F\big((x,0,0)\big) = \left(0,\frac{1}{6}x^2,0\right)$; for all $x \in \mathbb{R}^+$;
$F\big((0,y,0)\big) = \left(0,0,\frac{1}{6}y^2\right)$; for all $x \in \mathbb{R}^+$;
$F\big((0,0,z)\big) = \left(0,0,\frac{1}{6}z^2\right)$; for all $x \in \mathbb{R}^+$.

Let the function:$T: {\mathbb{R}^+}^n \to \mathbb{R}^+$ be defined by

$$\mathrm{T}(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - at_5t_6 - b\max\{t_2^2, t_3^2, t_4^2)$$

where $t_1 = d(F_x,F_y)$, $t_2 = d(x,y), t_3 = d(x,F_x)$, $t_4 = d\big(y,F_y\big)$, $t_5 = d\big(x,F_y\big) and t_6 = d(y,F_x)$, for all $x,y \in X$. Then $T \in \Phi$ and It is clear that $F$is a CCIRT for $a = \frac{1}{3}, b = \frac{1}{2}$. Therefore, all conditions of Theorem 3 are satisfied, and so F has a fixed point (which is $(0,0,0) \in A \cap B \cap C$).

4. Application To Nonlinear Volterra Integral Equations In Two Variables

Consider the nonlinear Volterra integral equations in two variables of the forms ([25]):

$$u(x,y) = f(x,y) + \int_0^x g\big(x,y,\xi,u(\xi,y)\big)d\xi + \int_0^x \int_0^y h\big(x,y,\sigma,\tau,u(\sigma,\tau)\big)d\tau d\sigma$$

where f,g,h are given functions and u is the unknown function to be found.

Let $\mathbb{R}$ denote the set of real numbers and $C(S_1,S_2)$ the class of continuous functionfrom the set S_1 to the S_2. We denote by $\mathbb{R}_+ = [0,\infty)$, $E = \mathbb{R}_+ \times \mathbb{R}_+$, $E_1 = \{f(x,y,s) : 0 \le s \le x < \infty, y \in \mathbb{R}+$; and $E2=\{fx,y,s,t : 0 \le s \le x < \infty, 0 \le t \le y < \infty\}$.

Throughout we assume that $f \in (E,\mathbb{R})$, $g \in C(E_1 \times \mathbb{R},\mathbb{R})$, $h \in (E_2 \times \mathbb{R},\mathbb{R})$.
Denote by S the space of functions $z = C(E_1,\mathbb{R})$ which fulfill the condition
(4.1) $\quad |z\,(x,t)| = \mathcal{O}(\exp(\lambda(x+y)))$,
where λ is a positive constant. Define the norm in the space S as
(4.2) $\quad |z|_S = sup_{(x,y)\in S}[|z(x,t)|\exp\,(-\lambda(x+y))]$.
It is easy to see that S with the norm defined in (4.2) is a Banach space. Wde note that the condition (4.1) implies that there is a constant $M_0 \ge 0$ such that $|z\,(x,t)| \le M_0 \exp\,(\lambda(x+y))$.
Using the fact in (4.2) we observe that
(4.3) $\quad |z|_S \le M_0$.
Define a mapping $F : S \to S$ by

$$Fu(x,y) = f(x,y) + \int_0^x g\big(x,y,\xi,u(\xi,y)\big)d\xi + \int_0^x \int_0^y h\big(x,y,\sigma,\tau,u(\sigma,\tau)\big)d\tau d\sigma$$

for $u \in S$. Note that, if $u^* \in S$ is fixed point of F. then u^* is asolution of the problem (4.1).
We shall prove the existence of a fixed point of F under the following conditions.

(I) There exist $(\alpha,\beta) \in S^2, (\alpha_0,\beta_0) \in \mathbb{R}^2$ such that

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$$\alpha_0(x,t) \leq \alpha(x,t) \leq \beta(x,t) \leq \beta_0(x,t) \in \mathbb{R}_+ \times \mathbb{R}_+$$

and for all $(x,t) \in \mathbb{R}_+ \times \mathbb{R}_+$, we have

$\alpha(x,t) \leq f(x,t) + \int_0^x g\big(t,s,\xi,\beta(\xi,s)\big)d\xi + \int_0^x \int_0^y h\big(t,s,\sigma,\tau,\beta(\sigma,\tau)\big)d\tau d\sigma$ and

$\beta(x,t) \leq f(x,t) + \int_0^x g\big(t,s,\xi,\alpha(\xi,s)\big)d\xi + \int_0^x \int_0^y h\big(t,s,\sigma,\tau,\alpha(\sigma,\tau)\big)d\tau d\sigma.$

(II) The function g, h in equation (4.1) satisfy the conditions

$$|g(x,y,\xi,u) - g(x,y,\xi,\bar{u}| \leq h_1(x,y,\xi)|u - \bar{u}|,$$

$|h(x,y,\sigma,\tau,u) - h(x,y,\sigma,\tau,\bar{u}| \leq h_2(x,y,\sigma,\tau)|u - \bar{u}|,$

where $h_1 \in C(E_1, \mathbb{R}_+), h_2 \in C(E_2, \mathbb{R}_+)$.

(III) There exist nonnegative constants $\delta_1 < 1$, δ_2 such that

$$\int_0^x h_1(x,y,\xi) \exp\big(\lambda(x+y)\big)\, d\xi + \int_0^x \int_0^y h_2(x,y,\sigma,\tau) \exp\,(\lambda(\sigma + \tau))\, d\tau d\sigma$$

$$\leq \delta_1 \exp\,(\,\lambda(x + y)),$$

and

$$\left| f(x,y) + \int_0^x g(x,y,\xi,0)d\xi + \int_0^x \int_0^y h(x,y,\sigma,\tau,0)d\tau d\sigma \right|$$

$$\leq \delta_2 \exp\,(\,\lambda(x + y)),$$

where λ is as given in (4.1).

(IV) The functions g, h in the equation (4.1) satisfy the conditions

u, v $\in \mathbb{R}$, u$\leq v \Rightarrow g(x,t,\xi,u) \geq g(x,t,\xi,v)$, for each $(x,y,\xi) \in E_1$, and

$h\big(x,t,\sigma,\tau,u(\sigma,\tau)\big) \geq h\big(x,t,\sigma,\tau,\beta(\sigma,\tau)\big)$ for each $(x,t,\sigma,\tau) \in E_2$.

(V) There exist $(\alpha,\beta) \in S^2$ such that $\alpha(t) \leq \beta(t)$ for $t \in \mathbb{R}_+$ and that

$(\mathcal{F}\alpha)(x,t) \leq \beta(x,t)$ and $(\mathcal{F}\beta)(x,t) \geq \alpha(x,t)$ for $(x,t) \in \mathbb{R}_+ \times \mathbb{R}_+$.

Theorem 4. *Underthe assumptions(I)-(III), the integral problem (4.1) has a unique solution* $u^* \in$ *Sand itbelongs to* $\mathcal{P} = \{u \in S: \alpha(x,y) \leq u(x,y) \leq \beta(x,y), (x,y) \in \mathbb{R}_+ \times \mathbb{R}_+\}$.

Proof. The proof of the theorem is divided into three parts.

(A) : First we show that $\mathcal{F}$ maps S into itself.

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Evidently, $\mathcal{F}u$is continuous on S and $\mathcal{F}u \in \mathbb{R}$. We verify that (4.1) is fulfilled. From (4.2), and using conditions (II), (III) and (4.3) we have

(4.4) $$|(\mathcal{F}u)(x,y)| \le \left|f(x,y) + \int_0^x g(x,y,\xi,0)d\xi + \int_0^x \int_0^y h(x,y,\sigma,\tau,0)d\tau d\sigma\right|$$

$$+ \int_0^x \left|g(x,y,\xi,u)(\xi,y)) - \int_0^x g(x,y,\xi,0)\right| d\xi$$
$$+ \int_0^x \int_0^y \left|h\big(x,y,\sigma,\tau,u(\sigma,\tau)\big) - h(x,y,\sigma,\tau,0)\right| d\tau d\sigma$$

$$\le \delta_2 exp\,(\,\lambda(x+\,y)) + \int_0^x h_1(x,y,\xi)|u(\xi,y)|d\xi$$

$$+ \int_0^x \int_0^y h_2(x,y,\sigma,\tau)|u(\sigma,\tau)|d\tau d\sigma$$

$$\le \delta_2 exp\,(\,\lambda(x+\,y))$$
$$+ |u|s\left[\int_0^x h_1(x,y,\xi)\exp\,(\lambda(x+\,y))d\xi + \int_0^x \int_0^y h_2(x,y,\sigma,\tau)\exp\,(\lambda(\sigma,\tau))d\tau d\sigma\right]$$

$$\le [\delta_2 + M_0\delta_1]\, exp\,(\,\lambda(x+\,y)).$$

It follows from (4.4) that $\mathcal{F}u \in S$. This proves that $\mathcal{F}$ maps S into itself.

(B) Defines closed subsets of S, $\mathcal{A}_1$ and $\mathcal{A}_2$ by
$\mathcal{A}_1 = \{u \in S: u(x,t) \le \beta(x,t)$, for $(x,t), \in \mathbb{R}_+ \times \mathbb{R}_+\}$
and
$\mathcal{A}_2 = \{u \in S: u(x,t) \ge \alpha(x,t)$ for $(x,t) \in \mathbb{R}_+ \times \mathbb{R}_+\}$.

We shall prove that

(4.5) $$F(A_1) \subseteq A_2 \text{ and } F(A_2) \subseteq A_1.$$

Let $u \in A_1$, that is $u(x,t) \le \beta(x,t) \in \mathbb{R}_+ \times \mathbb{R}_+$.
The condition (I), (IV) and (V) imply that

$$Fu(x,t) = f(x,t) + \int_0^x g\big(x,t,\xi,u(\xi,t)\big)d\xi + \int_0^x \int_0^y h\big(x,t,\sigma,\tau,u(\sigma,\tau)\big)d\tau d\sigma$$
$$\ge f(x,t) + \int_0^x g\big(x,t,\xi,\beta(\xi,t)\big)d\xi + \int_0^x \int_0^y h\big(x,t,\sigma,\tau,\beta(\sigma,\tau)\big)d\tau d\sigma \ge \alpha(x,t)$$

For all $(x,t) \in \mathbb{R}_+ \times \mathbb{R}_+$. Hence we have $Fu \in A_2$
Similarly, if $u \in A_2$, it can be proved that $Fu \in A_1$ holds. Thus (4.5) is fulfilled.
(C) : We verify that the operator F is a cyclic contracting mapping of an implicit relation type (3.13).

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Let $(u, v) \in A_1 \times A_2$, that is for all $t \in J$
$u(x,t) \leq \beta(x,t) \leq \beta_0, v\,(x,t) \leq \alpha(x,t) \geq \alpha_{0.}$
Using the properties (4.4) of F and condition (II) and (III), we conclude that
(4.6) $|(Fu)(x,y) - (Fv)(x,y)|$

$$\leq \int_0^x \left|g\big(x,y,\xi,u(\xi,y)\big) - g\big(x,y,\xi,v(\xi,y)\big)\right| d\xi$$

$$+ \int_0^x \int_0^y \left|h\big(x,y,\sigma,\tau,u(\sigma,\tau)\big) - h\big(x,y,\sigma,\tau,u(\sigma,\tau)\big)\right| d\tau d\sigma$$

$\leq \int_0^x h_1\,(x,y,\xi)|u(\xi,y) - v(\xi,y)|d\xi + \int_0^x \int_0^y h_2\ \,(x,y,\sigma,\tau|u(\sigma,\tau) - v(\sigma,\tau)|d\tau d\sigma$

$$\leq |u - v|_S \left[\int_0^x h_1\,(x,y,\xi)\exp\,(\lambda(x+y))d\xi\ + \int_0^x \int_0^y h_2\ \,(x,y,\sigma,\tau)\exp\,(\lambda(\sigma,\tau))\ d\tau d\sigma\right]$$

$$\leq \delta_1 |u - v|_S \exp\,(\lambda(x+y).$$

This implies that

$$|Fu - Fv|_S \leq b|u - v|_S$$

where $b = \delta^2 < 1$. Hence F satisfy condition (3.14) when $a, b, c, d = 0, b < 1$ such that $a + b + c + d < 1$.

Using the same technique, we can show that the above inequality also holds if we take $(u, v) \in A_{2\times} A_{1.}$
All other condition of theorem 2 are fulfilled for the complete metric space $(A_1 \cup A_2, ||s)$ and F restricted to $A_1 \cup A_2$ (with p = 2).
We conclude that operator F has a unique fixed point u^* and, hence, the integrodifferential equation (4.1) has a unique solution in the set P.

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