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Existence Theory for Third Order Random Differential Inclusion

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Abstract
In this paper, we prove the existence result for solution of the initial value problem of third order random differential inclusion through random fixed point theory using caratheodory condtion . We claim that our result is new to the theory of random differential inclusion.

Keywords: Random differential inclusion, random solution, Caratheodory condition.

AMS Mathematics Subject Classifications: 60H25, 47H10.

1. Statement of the Problem

Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ-finite measure space and let R be the real and and let $J = [0, T]$ be a closed and bounded interval in R. Consider the initial value problem of third order ordinary random differential inclusion (in short RDI),

$$\left.\begin{array}{l} x'''(t,\omega) \in F(t, x(t,\omega), \omega) + G(t, x(t,\omega), \omega) \quad \text{a.e.} \quad t \in J \\ x(0,\omega) = q_0(\omega), x'(0,\omega) = q_1(\omega), x''(0,\omega) = q_2(\omega) \end{array}\right\} \tag{1.1}$$

for all $\omega \in \Omega$, where $q_0, q_1, q_2 : \Omega \to R$ is measurable and $F, G : J \times R \times \Omega \to \mathcal{P}_p(R)$.

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By a random solution for the RDI (1.1) we mean a measurable function $x:\Omega \to AC(J,R)$ satisfying for each $\omega \in \Omega$,

$$x'''(t,\omega) = v_1(t,\omega) + v_2(t,\omega) \ \forall t \in J .$$

and $x(0,\omega) = q_0(\omega), x'(0,\omega) = q_1(\omega), x''(0,\omega) = q_2(\omega)$ for some measurable functions $v_1, v_2 : \Omega \to L^1(J,R)$ with $v_1(t,\omega) \in F(t, x(t,\omega), \omega)$ a.e. $t \in J$ and $v_2(t,\omega) \in G(t, x(t,\omega), \omega)$ a.e. $t \in J$, where $AC(J,R)$ is the space of absolutely continuous real-valued functions on *J*.

The RDI (1.1) has not been discussed earlier in the literature. In this paper, we discuss and prove the existence of solution for the initial value problem of third order random differential inclusion through random fixed point theory.

2. Auxiliary Results

Theorem 2.1 (Kuratowskii and Ryll-Nardzewski [11]) If the multi-valued operator $T:\Omega \times X \to \mathcal{P}_p(X)$ is measurable with closed values, then T has a measurable selector.

Random fixed point theorem for the right monotone increasing condensing multi-valued random operators is as follows :

Theorem 2.2. Let $(\Omega, \mathcal{A})$ be a measurable spaces and let $[a,b]$ be a random interval in a separable Banach space X. If $T:\Omega \times [a,b] \to \mathcal{P}_{cl}([a,b])$ is a condensing, upper semi-continuous right monotone increasing multi-valued random operator and the cone *K* in *X* is normal, then $T(\omega)$ has a random fixed point in $[a,b]$.

Corollary 2.1 (Dhage [2]) Let $(\Omega, \mathcal{A})$ be a measurable space and let $[a,b]$ be a random interval in a separable Banach space X. If $T:\Omega \times [a,b] \to \mathcal{P}_{cl}([a,b])$ is a compact, upper semi-continuous right monotone increasing multi-valued random operator and the cone K in X is normal, then $T(\omega)$ has a random fixed point in $[a,b]$.

To prove the next result, we need the following lemmas.

Lemma 2.1(Dhage [4]) Let $(\Omega, \mathcal{A}, \mu)$ be a complete $\sigma -$ *finite* measure space and let X be separable Banach space. If $F, G : \Omega \to \mathcal{P}_{cl}(X)$ are two multi-valued random operators, then the sum F + G defined by $(F+G)(\omega)=F(\omega)+G(\omega)$ is again a multi-valued random operator on Ω.

Corollary 2.2. Let $(\Omega,\mathcal{A},\mu)$ be a complete σ-finite measure space and let $[a,b]$ be a random interval in a separable Banach space X. Let $A,B:\Omega\times[a,b]\to\mathcal{P}_{cl}(X)$ be two right monotone increasing multi-valued random operators satisfying for each $\omega\in\Omega$,

(a) $A(\omega)$ is a multi-valued contraction,

(b) $B(\omega)$ is completely continuous, and

(c) $A(\omega)x+B(\omega)x\in[a,b]$ for all $x\in[a,b]$.

Furthermore, if the cone K in X is normal, then the random operator inclusion $x\in A(\omega)x+B(\omega)x$ has a random solution in $[a,b]$.

Corollary 2.3 : Let $(\Omega,\mathcal{A},\mu)$ be a complete $\sigma-$finite measure space and let $[a,b]$ be a random interval in a separable Banach space X. Assume that $A:\Omega\times[a,b]\to X$ is nondecreasing and $B:\Omega\times[a,b]\to\mathcal{P}_{cl}(X)$ is a right monotone increasing multi-valued random operator satisfying for each $\omega\in\Omega$,

(a) $A(\omega)$ is a single-valued contraction with the contraction constant $\lambda<1/2$,

(b) $B(\omega)$ is completely continuous, and

(c) $A(\omega)x+B(\omega)x\in[a,b]$ for all $x\in[a,b]$.

Furthermore, if the cone K in X is normal, then the random operator inclusion $x\in A(\omega)x+B(\omega)x$ has a random solution in $[a,b]$.

3. Exitence Results

Let $S^1_{F(\omega)}(x)=\left\{v\in\mathcal{M}\big(\Omega,L^1(J,R)\big)\,\middle|\,v(t,\omega)\in F(t,x(t,\omega),\omega)\ \text{a.e. } t\in J\right\}$. (3.1)

This is our set of selection functions. The integral of the random multi-valued function *F* is defined as

$$\int_0^t F(s,x(s,\omega),\omega)\,ds=\left\{\int_0^t v(s,\omega)\,ds : v\in S^1_{F(\omega)}(x)\right\}.$$

We need the following definitions.

Definition **3.1.**A multi-valued function $F : J \times R \times \Omega \to \mathcal{P}_{cp}(R)$ is called Caratheodory if for each $\omega \in \Omega$,

(i) $t \mapsto F(t,x,\omega)$ is measurable for each $x \in R$, and

(ii) $x \mapsto F(t,x,\omega)$ is an upper semi-continuous almost everywhere for $t \in J$.

Again, a Caratheodory multi-valued function F is called L^1-Caratheodory if

(iii) for each real number r > 0 there exists a measurable function $h_r : \Omega \to L^1(J,R)$ such that for each $\omega \in \Omega$

$$\|F(t,x,w)\|_{\mathcal{P}} = \sup\{|u| : u \in F(t,x,\omega)\} \le h_r(t,\omega) \text{ a.e. } t \in J$$

for all $x \in R$ with $|x| \le r$.

Furthermore, a Caratheodory multi-valued function F is called L^1_X-Caratheodory if

(iv) there exists a measurable function $h : \Omega \to L^1(J,R)$ such that

$$\|F(t,x,\omega)\|_{\mathcal{P}} \le h(t,\omega) \text{ a.e. } t \in J$$

for all $x \in R$,and the function h is called a growth function of F on $J \times R \times \Omega$.

Definition **3.2**.A multi-valued function $F : J \times R \times \Omega \to P_{cp}(R)$ is called s-Caratheodory if for each $\omega \in \Omega$,

(i) $(t,\omega) \mapsto F(t,x,\omega)$ is measurable for each $x \in R$, and

(ii) $x \mapsto F(t,x,\omega)$ is an Hausdorff continuous almost everywhere for $t \in J$.

Furthermore, a s-Caratheodory multi-valued function F is called s-L^1- Caratheodory if

(iii) for each real number r > 0 there exists a measurable function $h_r : \Omega \to L^1(J,R)$ such that for each $w \in \Omega$

$$\|F(t,x,\omega)\|_{\mathcal{P}} = \sup\{|u| : u \in F(t,x,\omega)\} \le h_r(t,\omega) \text{ a.e. } t \in J$$

for all $x \in R$ with $|x| \le r$.

Then we have quote the following lemmas.

Lemma 3.1 (Lasota and Opial [8]) Let E be a Banach space. If dim (E) < ∞ and $F: J\times E\times\Omega\to\mathcal{P}_{c\mathcal{P}}(E)$ is L^1- Caratheodory, then $S^1_{F(\omega)}(x)\neq\emptyset$ for each $x\in E$.

Lemma 3.2 (Lasota and Opial [8]) Let E be a Banach space, F a Caratheodory multi-valued operator with $S^1_{f(\omega)}\neq\emptyset$, and $\ell: L^1(J,E)\to C(J,E)$ be a continuous linear mapping. Then the composite operator

$$\mathcal{L}\circ S^1_{F(\omega)}: C(J,E)\to\mathcal{P}_{bd,cl}(C(J,E))$$

is a closed graph operator on $C(J,E)\times C(J,E)$.

Lemma3.3 (Caratheodory theorem[5]) Let E be a Banach space. If $F: J\times E\to\mathcal{P}_p(E)$ is s-Caratheodory, then the multi-valued mapping $(t,x)\mapsto F(t,x(t))$ is jointly measurable for each measurable function $x: J\to E$.

We consider the following set of hypotheses .

(A_1) The multi-valued mapping $(t,\omega)\mapsto F(t,x,\omega)$ is jointly measurable for all $x\in R$.

(A_2) $F(t,x,w)$ is closed and bounded for each $(t,\omega)\in J\times\Omega$ and $x\in R$.

(A_3) F is integrably bounded on $J\times\Omega\times R$.

(A_4) There is a function $\ell\in\mathcal{M}(\Omega, L^1(J,R))$ such that for each $\omega\in\Omega$,

$$d_H(F(t,x,\omega),F(t,y,\omega))\leq\ell(t,\omega)|x-y| \quad \text{a.e. } t\in J$$

for all $x,y\in R$.

(A_5) The multi-valued mapping $x\mapsto S^1_{F(\omega)}(x)$ is right monotone increasing in $x\in C(J,R)$ almost everywhere for $t\in J$ and

(B_1) The multi-valued mapping $(t,\omega)\mapsto G(t,x(t,\omega),\omega)$ is jointly measurable for all measurable $x:\Omega\to C(J,R)$.

(B_2) $G(t,x,\omega)$ is closed and bounded for each $(t,\omega)\in J\times\Omega$ and $x\in R$.

(B_3) G is L^1-Caratheodory.

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(B_4) The multi-valued mapping $x \mapsto S^1_{F(\omega)}(x)$ is right monotone increasing in $x \in C(J,R)$ almost everywhere for $t \in J$.

(B_5) RDI (1.1) has a strict lower random solution a and a strict upper random solution b with $a \le b$ on $J \times \Omega$.

Main Result

Theorem3.1 Assume that the hypotheses $(A_1)-(A_5)$ and $(B_1)-(B_5)$ hold. If $\|\ell(\omega)\|_{L^1} < 1$, then the RDI (1.1) has a random solution in [a, b] defined on $J \times \Omega$ Proof : Let $X = C(J,R)$. Define a random order interval $[a,b]$ in X which is well defined in view of hypothesis (B_5). Now the RDI (1.1) is equivalent to the random integral inclusion,

$$x'''(t,\omega) \in q_0(\omega) + q_1(\omega)\omega + q_2(\omega)\frac{\omega^2}{2} + \int_0^t \frac{(t-s)^2}{2} F(s, x(s,\omega), \omega)\, ds + \int_0^t \frac{(t-s)^2}{2} G(s, x(s,\omega), \omega)\, ds, \quad t \in J. \tag{3.2}$$

for all $\omega \in \Omega$. Define two multi-valued operators $A, B : \Omega \times [a,b] \to \mathcal{P}_{\mathcal{P}}(X)$ by

$$A(\omega)x = \left\{ u \in \mathcal{M}(\Omega, X) \mid u(t,\omega) = \int_0^t v_1(s,\omega)\, ds, v_1 \in S^1_{F(\omega)}(x) \right\} \tag{3.3}$$

$$= \left(\mathcal{K}_1 \circ S^1_{F(\omega)} \right)(x)$$

And

$$B(\omega)x = \left\{ u \in \mathcal{M}(\Omega, X) \mid u(t,\omega) = q_0(\omega) + q_1(\omega)\omega + q_2(\omega)\frac{\omega^2}{2} + \int_0^t v_2(s,\omega)\, ds, v_2 \in S^1_{F(\omega)}(x) \right\} \tag{3.4}$$

$$= \left(\mathcal{K}_2 \circ S^1_{F(\omega)} \right)(x)$$

where $\mathcal{K}_1, \mathcal{K}_2 : \mathcal{M}(\Omega, L^1(J,R)) \to \mathcal{M}(\Omega, C(J,R))$ are continuous operators defined by

$$\mathcal{K}_1 v_1(t,\omega) = \int_0^t v_1(s,\omega)\, ds, \tag{3.5}$$

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and $\mathcal{K}_2 v_2(t,\omega) = q_0(\omega) + q_1(\omega)\omega + q_2(\omega)\frac{\omega^2}{2} + \int_0^t v_2(s,\omega)ds.$ (3.6)

Clearly, the operators $A(\omega)$ and $B(\omega)$ are well defined in view of hypotheses (A_3) and (B_3). We will show that $A(\omega)$ and $B(\omega)$ satisfy all the conditions of Corollary 2.2.

Step I : First, we show that A is closed valued multi-valued random operator on $\Omega \times [a,b]$. Observe that the operator $A(\omega)$ is equivalent to the composition $\mathcal{K}_1 \circ S^1_{F(w)}$ of two operators on $L^1(J,R)$, where $\mathcal{K}_1 : \mathcal{M}(\Omega, L^1(J,R)) \to X$ is the continuous operator . To show $A(\omega)$ has closed values, it then suffices to prove that the composition operator $\mathcal{K}_1 \circ S^1_{F(\omega)}$ has closed values on [a, b]. Let $x \in [a,b]$ be arbitrary and let $\{v_n\}$ be a sequence in $S^1_{F(\omega)}(x)$ converging to v in measure. Then, by the definition of $S^1_{F(\omega)}$, $v_n(t,\omega) \in F(t,x(t,\omega),\omega)$ a.e. for $t \in J$. Since $F(t,x(t,\omega),\omega)$ is closed, $v(t,\omega) \in F(t,x(t,\omega),\omega)$ a.e. for $t \in J$. Hence, $v \in S^1_{F(\omega)}(x)$. As a result, $S^1_{G(\omega)}(x)$ is a closed set in $L^1(J,R)$ for each $\omega \in \Omega$. From the continuity of $\mathcal{K}_1$, it follows that $(\mathcal{K}_1 \circ S^1_{F(\omega)})(x)$ is a closed set in X. Therefore, $A(\omega)$ is a closed-valued multi-valued operator on [a, b] for each $\omega \in \Omega$. Next, we show that $A(\omega)$ is a multi-valued random operator on [a, b]. First, we show that the multi-valued mapping $(\omega, x) \mapsto S^1_{F(\omega)}(x)$ is measurable. Let $f \in \mathcal{M}(\Omega, L^1(J,R))$ be arbitrary. Then we have

$$d\left(f, S^1_{F(\omega)}(x)\right) = \inf\left\{\|f(\omega) - h(\omega)\|_{L^1} : h \in S_{F(\omega)}(x)\right\}$$

$$= \int_0^T d\left(f(t,\omega)\right), F(t,x(t,\omega),\omega)dt.$$

But by hypothesis (A_1), the mapping $(t,\omega) \mapsto F(t,x,\omega)$ is measurable, and by (A_4), the mapping $x \mapsto F(t,x,\omega)$ is Hausdorff continuous. Hence by Caratheodory theorem, the map $(t,\omega) \mapsto F(t,x(t,\omega),\omega)$ is measurable for all measurable function $x : \Omega \to C(J,R)$. It is known that the multi-valued mapping $z \mapsto d(z, F(t,x,\omega))$ is continuous, Hence the mapping multi-valued mapping $(t,x,\omega,z) \mapsto d(z, F(t,x,\omega))$ measurable. Hence we deduce that the mapping (t,x,ω,f)

$\mapsto d\big(f(t,\omega),F(t,x(t,\omega),\omega)\big)$ is measurable from $J\times X\times\Omega\times L^1(J,R)$ into R^+. Now the integral is the limit of the finite sum of measurable functions, and so, $d\left(f,S^1_{F(\omega)}(x)\right)$ is measurable. As a result, the multi-valued mapping $(.,.)\rightarrow S^1_{F(\cdot)}(\cdot)$ is jointly measurable.

Define a function ϕ on $J\times X\times\Omega$ by

$$\phi(t,x,\omega)=\left(\mathcal{K}_1 S^1_{F(\omega)}\right)(x)(t)=\int_0^t F(s,x(s,\omega,\omega))\,ds.$$

We shall show that $\phi(t,x,\omega)$ is continuous in t in the Hausdorff metric on R. Let $\{t_n\}$ be a sequence in converging to $t\in J$. Then

$$\begin{aligned}
&d_H\left(\phi(t_n,x,\omega),\phi(t,x,\omega)\right)\\
&=d_H\left(\int_0^{t_n}F(s,x(s,\omega),\omega)\,ds,\int_0^{t}F(s,x(s,\omega),\omega)\,ds\right)\\
&=\int_J\left|\mathcal{X}_{[0,t_n]}(s)-\mathcal{X}_{[0,t]}(s)\right|h_r(s,\omega)\,ds\\
&\rightarrow 0 \quad \text{as} \quad n\rightarrow\infty.
\end{aligned}$$

Thus the multi-valued mapping $t\mapsto\phi(t,x,\omega)$ is continuous, and hence, and by Lemma 3.3, the mapping $(t,x,\omega)\mapsto\int_0^t F(s,x(s,\omega),\omega)\,ds$ is measurable. Consequently, $A(\omega)$ is a random multi-valued operator on [a, b]. Similarly, it can be shown that $B(\omega)$ is a closed-valued multi-valued operator on [*a, b*] and the mapping $(t,x,\omega)\mapsto\int_0^t G(s,x(s,\omega),\omega)\,ds$ is measurable. Again, since the sum of two measurable multi-valued functions is measurable, the mapping $(t,x,\omega)\mapsto$ $q_0(\omega)+q_1(\omega)\omega+q_2(\omega)\frac{\omega^2}{2}+\int_0^t G(s,x(s,\omega),\omega)\,ds$ is measurable.

Step II : Next we show that $A(\omega)$ is a multi-valued contraction on X. Let $x,y\in X$ be any two element and let $u_1\in A(\omega)(x)$. Then $u_1\in X$ and $u_1(t,\omega)=\int_0^t v_1(s,\omega)\,ds$ for some $v_1\in S^1_{F(\omega)}(x)$. Since

$$d_H\big(F(t,x(t,\omega),\omega),F(t,y(t,\omega),\omega)\big)\le\ell(t,\omega)\left|x(t,\omega)-y(t,\omega)\right|,$$

we obtain that there exists a $w \in F\left(t, y\left(t,\omega\right),\omega\right)$ such that

$$\left|v_1\left(t,\omega\right) - w\right| \le \ell\left(t,\omega\right)\left|x\left(t,\omega\right) - y\left(t,\omega\right)\right|.$$

Thus, the multi-valued operator U defined by

$$U\left(t,\omega\right) = S^1_{F(\omega)}\left(y\right)\left(t\right) \cap K\left(\omega\right)\left(t\right),$$

where $K\left(\omega\right)\left(t\right) = \left\{w \left| v_1\left(t,\omega\right) - w\right| \le \ell\left(t,\omega\right)\left|x(t,\omega) - y\left(t,\omega\right)\right|\right\}$

has nonempty values and is measurable. Let v_2 be a measurable selection function for U. Then there exists $v_2 \in F\left(t, y\left(t,\omega\right),\omega\right)$ with
$\left|v_1\left(t,\omega\right) - v_2\left(t,\omega\right)\right| \le \ell\left(t,\omega\right)\left|x\left(t,\omega\right) - y\left(t,\omega\right)\right|$, a.e. on J.

Define $u_2\left(t,\omega\right) = \int_0^t v_2\left(s,\omega\right)ds.$ It follows that $u_2 \in A\left(\omega\right)\left(y\right)$ and

$$\begin{aligned}\left|u_1\left(t,\omega\right) - u_2\left(t,\omega\right)\right| &\le \left|\int_0^t v_1\left(s,\omega\right)ds - \int_0^t v_2\left(s,\omega\right)ds\right| \\ &\le \int_0^t \left|v_1\left(s,\omega\right) - v_2\left(s,\omega\right)\right|ds \\ &\le \left\|\ell\left(\omega\right)\right\|_{L^1}\left\|x\left(\omega\right) - y\left(\omega\right)\right\|.\end{aligned}$$

Taking the supremum over t, we obtain

$$\left\|u_1\left(\omega\right) - u_2\left(\omega\right)\right\| \le \left\|\ell\left(\omega\right)\right\|_{L^1}\left\|x\left(\omega\right) - y\left(\omega\right)\right\|.$$

From this and the analogous inequality obtained by interchanging the role of x and y we obtain

$$d_H\left(A(\omega)(x), A(\omega)(y)\right) \le \left\|\ell(\omega)\right\|_{L^1}\left\|x(\omega) - y(\omega)\right\|,$$

for all $x, y \in X$. This shows that $A\left(\omega\right)$ is multi-valued random contraction on X, since $\left\|\ell\left(\omega\right)\right\|_{L^1} < 1$ for each $\omega \in \Omega$.

<u>Step</u> III : Next, we show that $B(\omega)$ is completely continuous for each $\omega\in\Omega$. First, we show that $B(\omega)([a,b])$ is compact for each $\omega\in\Omega$. Let $\{y_n(\omega)\}$ be a sequence in $B(\omega)([a,b])$ for some $\omega\in\Omega$. We will show that $\{y_n(\omega)\}$ has a cluster point. This is achieved by showing that $\{y_n(\omega)\}$ is uniformly bounded and equi-continuous sequence in X.

<u>Case</u> I : First, we show that $\{y_n(\omega)\}$ is uniformly bounded sequence. By the definition of $\{y_n(\omega)\}$, we have a $v_n(\omega)\in S^1_{G(\omega)}(x)$ for some $x\in[a,b]$ such that

$$y_n(t,\omega)=q_0(\omega)+q_1(\omega)\omega+q_2(\omega)\frac{\omega^2}{2}+\int_0^t v_n(s,\omega)ds,\quad t\in J.$$

Therefore,

$$\begin{aligned}|y_n(t,\omega)|&\le\left|q_0(\omega)+q_1(\omega)\omega+q_2(\omega)\frac{\omega^2}{2}\right|+\int_0^t|v_n(s,\omega)|ds\\&\le\left|q_0(\omega)+q_1(\omega)\omega+q_2(\omega)\frac{\omega^2}{2}\right|+\|h_r(\omega)\|_{L^1}\end{aligned}$$

for all $t\in J$, where $r=\|a(\omega)\|+\|b(\omega)\|$. Taking the supremum over t in the above inequality yields,

$$\|y_n(\omega)\|\le\left|q_0(\omega)+q_1(\omega)\omega+q_2(\omega)\frac{\omega^2}{2}\right|+\|h_r(\omega)\|_{L^1}$$

which shows that $\{y_n(\omega)\}$ is a uniformly bounded sequence in $Q(\omega)([a,b])$.

Next we show that $\{y_n(\omega)\}$ is an equi-continuous sequence in $Q(\omega)([a,b])$. Let $t,\tau\in J$. Then we have

$$\begin{aligned}|y_n(t,\omega)-y_n(\tau,\omega)|&\le\left|\int_0^t v_n(s,\omega)ds-\int_0^\tau v_n(s,\omega)ds\right|\\&\le|p(t,\omega)-p(\tau,\omega)|,\end{aligned}$$

where $p(t,\omega)=\int_0^t h_r(s,\omega)ds.$ From the above inequality, it follows that

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$$\left|y_n(t,\omega)-y_n(\tau,\omega)\right|\to 0 \quad \text{as} \quad t\to\tau .$$

This shows that $\{y_n(\omega)\}$ is an equi-continuous sequence in $B(\omega)([a,b])$. Now $\{y_n(\omega)\}$ is uniformly bounded and equi-continuous for each $\omega\in\Omega$, so it has a cluster point in view of Arzela-Ascoli theorem. As a result, $B(\omega)$ is a compact multi-valued random operator on [*a, b*].

Next we show that $B(\omega)$ is a upper semi-continuous multi-valued random operator on [*a, b*]. Let $\{x_n(\omega)\}$ be a sequence in *X* such that $x_n(\omega)\to x_*(\omega)$. Let $\{y_n(\omega)\}$ be a sequence such that $y_n(\omega)\in B(\omega)x_n$ and $y_n(\omega)\to y_*(\omega)$. We will show that $y_*(\omega)\in B(\omega)x_*$. Since $y_n(\omega)\in B(\omega)x_n$, there exists a $v_n(\omega)\in S^1_{G(\omega)}(x_n)$ such that

$$y_n(t,\omega)=q_0(\omega)+q_1(\omega)\omega+q_2(\omega)\frac{\omega^2}{2}+\int_0^t v_n(s,\omega)ds,\ \ t\in J\cdot$$

We must prove that there is a $v_*(\omega)\in S^1_{G(\omega)}(x_*)$ such that

$$y_*(t,\omega)=q_0(\omega)+q_1(\omega)\omega+q_2(\omega)\frac{\omega^2}{2}+\int_0^t v_*(s,\omega)ds,\ \ t\in J.$$

Consider the continuous linear operator $\mathcal{L}:\mathcal{M}(\Omega,L^1(J,R))\to\mathcal{M}(\Omega,C(J,R))$ defined by

$$\mathcal{L}v(t,\omega)=\int_0^t v(s,\omega)ds, \qquad t\in J.$$

Now

$$\left\|\left(y_n(\omega)-(q_0(\omega)+q_1(\omega)\omega+q_2(\omega)\frac{\omega^2}{2}q(\omega))\right)-\left(y_*(\omega)-(q_0(\omega)+q_1(\omega)\omega+q_2(\omega)\frac{\omega^2}{2})\right)\right\|\to 0 \quad as \quad n\to\infty$$

From lemma 3.2, it follows that $\mathcal{L}\circ S^1_{G(\omega)}$ is a closed graph operator. Also, from the definition of $\mathcal{L}$, we have

$$y_n(t,\omega)-(q_0(\omega)+q_1(\omega)\omega+q_2(\omega)\frac{\omega^2}{2})\in\left(\mathcal{L}\circ S^1_{G(\omega)}\right)(x_n)$$

Since $y_n(\omega)\to y_*(\omega)$, there is a point $v_*(\omega)\in S^1_{F(\omega)}(x_*)$ such that

$$y_*(t,\omega)=(q_0(\omega)+q_1(\omega)\omega+q_2(\omega)\frac{\omega^2}{2})+\int_0^t v_*(s,\omega)ds,\qquad t\in J.$$

This shows that $B(\omega)$ is a upper semi-continuous multi-valued random operator on [a, b].

Thus, $B(\omega)$ is upper semi-continuous and compact and hence a completely continuous multi-valued random operator on [*a*, *b*].

Step VI : Next, we show that $A(\omega)$ is a right monotone increasing and multi-valued random operator on [a, b] into itself for each $\omega\in\Omega$. Let $x,y\in[a,b]$ be such that $x\le y$. Since (A_5) holds, we have that $S^1_{F(\omega)}(x)\le S^1_{F(\omega)}(y)$. Hence $A(\omega)(x)\overset{i}{\le}A(\omega)(y)$.Similarly, $B(\omega)(x)\overset{i}{\le}B(\omega)(y)$. From (B_5), it follows that $a\le A(\omega)a+B(\omega)a$ and $A(\omega)b+B(\omega)b\le b$ for all $\omega\in\Omega$. Now $A(\omega)$ and $B(\omega)$ are monotone increasing, so we have for each $\omega\in\Omega$,

$$a\le A(\omega)a+B(\omega)a\overset{i}{\le}A(\omega)x+B(\omega)x\overset{i}{\le}A(\omega)b+B(\omega)b\le b$$

for all $x\in[a,b]$. Hence, $A(\omega)x+B(\omega)x\in[a,b]$ for all $x\in[a,b]$.

Thus, the multi-valued random operators $A(\omega)$ and $B(\omega)$ satisfy all the conditions of Corollary 2.2 and hence the random operator inclusion $x\in A(\omega)x+B(\omega)x$ has a random solution. This implies that the RDI (4.3.1) has a random solution on $J\times\Omega$. This completes the proof.

References

[1]. Aubin J. and A. Cellina, *Differential Inclusions*, Springer-Verlag, 1984.

[2]. B. C. Dhage, *Monotone increasing multi-valued random operators and differential inclusions*, Nonlinear Funct. Anal. and Appl. 12(3), 2007 .

[3]. B. C. Dhage, *Some algebraic fixed point theorems for multi- valued operators with applications*, Discuss. Math. Diff. Incl. Control and Optim. 26, 2006, 5-55.

[4]. B.C.Dhage, *Monotone method for discontinuous differential inclusions,* Math. Sci. Res. J. 8 (3), 2004, 104-113.

[5]. S.S. Bellale and S.B. Birajdar, *On Existence Theorem for Nonlinear Functional Two Point Boundary Value Problems in Banach Algebras,* Int. Jr. Universal Mathematics and Mathematical Sciences 1(1) (2015), 47-58.

[6]. S.S. Bellale, S.B. Birajdar and D.S. Palimkar, *Existence Theorem for Abstract Measure Delay Integro-Differential Equations,* Applied and Computational Mathematics 4(4) (2015), 225-231, doi:10.11648/j.acm.20150404.11.

[7]. S.S. Bellale, S.B. Birajdar, *Approximating Positive Solutions of Non-Linear Two Point Functional Boundary Value of Second Order Differential Equations,* International Journal of Scientific & Engineering Research, Volume 6, (6) 2015, 707-712.

[8]. S.B. Nimse and A.M. Burande, *Importance and Interaction of Osteoporotic Eractures in Indian Women by Using Fuzzy Measure, International Journal of Universal Science and Technology,* 01 (01) (2015), 01-10.

[9]. S. Hu and N.S. Papageorgiou, *Hand Book of Multi-valued Analysis Vol.-I, Kluwer Academic Publisher,* Dordrechet, Boston, London, 1997.

[10]. S. Itoh, *Random fixed point theorems with applications to random differential equations in Banach spaces,* J. Math. Anal. Appl. **67**, 1979, 261-273.

[11]. K. Kuratowskii and C. Ryll-Nardzeuskii, *A general theorem on selectors,* Bull. Acad. Pol.Sci. Ser. Math. Sci. Astron. Phy. **13**, 1965, 397-403.

[12]. A. Lasota and Z. Opial, *An application of Kakutani-Ky Fan theorem in the theory of ordinary differential equations,* Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phy. 13, 1965, 781-786.

[13]. D.S.Palimkar, *Existence Theory for Second Order Random Differential Inclusion, International Journal of Advances in Engineering,* Science and Technology, Vol. 2, No. 3, 2012.