International Journal of Mathematics and Mathematical Sciences
ISSN: 2454-7271, Copyright@universalprint
Volume: 01 No.: 02, Pages: 59-69 , Published:Dec.,2015
Web: www.universalprint.org , Publication: Universal Print
Title Key: Some Fixed Point Results for Single and Two Maps…

Some Fixed Point Results for Single and Two Maps in Complete Metric Space

Dinesh Panthi[1], P. Sumati Kumari[2]
[1]Department of Mathematics Valmeeki Campus
Nepal Sanskrit University, Nepal
Email: panthid06@gmail.com

[2]Department of Mathematics, KL University, Vaddeswaram
A. P. , India
Email: mumy143143143@gmail.com

Abstract
In this article we establish some fixed point theorems for singe and two mappings in complete metric space which generalize and extend some similar results in the literature.

Keywords: metric space, continuous function, cauchy sequence, fixed point

Subject Classification: 47H10, 54H25.

1. Introduction

In 1922, S. Banach [11] established a fixed point theorem for contraction mapping in complete metric space which is famous as Banach Fixed Point Theorem. Since then, many generalizations of this theorem have been obtained by various authors by weakening its hypothesis while retaining the convergence property of successive iterates to the unique fixed point of the mapping.
The purpose of this article is to establish some fixed point results for single and pair of mappings which generalize and extend some existing well-known results in the literature.
Now we start with following definitions, lemmas and theorems.

2. Preliminaries

Definition 1 Let X be a non empty set and d be a real function from $X \times X$ into $\mathbb{R}^+$ such that for all $x, y, z \in X$, we have

1. $d(x, y) \geq 0$

International Journal of Mathematics and Mathematical Sciences
ISSN: 2454-7271, Copyright@universalprint
Volume: 01 No.: 02, Pages: 59-69 , Published:Dec.,2015
Web: www.universalprint.org , Publication: Universal Print
Title Key: Some Fixed Point Results for Single and Two Maps…

2. $d(x,y)=0 \Leftarrow x=y$
3. $d(x,y)=d(y,x)$
4. $d(x,z)\le d(x,y)+d(y,z)$

then, d is called a **metric** or distance function and the pair (X,d) is called a **metric space**.

Definition 2 *A sequence* $\{x_n\}$ *in a metric space* (X,d) *is called a Cauchy sequence if for given* ε *> 0, there corresponds* $n_0 \in N$ *such that for all* $m,n \ge n_0$ *, we have* $d(x_m,x_n) < \varepsilon$ *.*

Definition 3 *A sequence in metric space converges with respect to d (or in d) if there exists* $x \in X$ *such that* $d(x_n,x) \to 0$ *as* $n \to \infty$.

In this case, x is called limit of $\{x_n\}$ and we write $x_n \to x$.

Lemma 1 *Every subsequence of a convergent sequence to a point* x_0 *is convergent* x_0.

Theorem 1 (Banach's Contraction Mapping Theorem) Let (X, d) be a complete metric space and $T: X \to X$ be a map such that $d(Tx,Ty) \le \alpha d(x,y)$ for some $0 \le \alpha < 1$ and all $x,y \in X$ then T has a unique fixed point in X.

Moreover, for any $x_0 \in X$ the sequence of Picard iterates $\{T^n x_0\}, n \ge 0$ converges to the fixed point of T.

3 Main Results

Theorem 2 *Let (X, d) be a complete metric space. Let* $T: X \to X$ *be continuous mapping satisfying the condition*

$$d(Tx,Ty) \le \alpha \frac{d(x,Tx)d(y,Ty)+d(x,Ty)d(y,Tx)}{d(x,y)} + \beta \frac{d(x,Ty)[d(x,Tx)+d(y,Ty)]}{d(x,y)+d(y,Ty)+d(y,Tx)} \quad (1)$$

$$+\gamma \frac{d(x,Tx)d(y,Tx)+d(y,Ty)d(x,Ty)}{d(x,Tx)+d(y,Tx)+d(y,Ty)+d(x,Ty)} + \kappa \frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{d(x,Ty)+d(y,Tx)}$$
$$+\delta[d(x,Tx)+d(y,Ty)]+\eta[d(y,Tx)+d(x,Ty]+\mu d(x,y)$$

for all $x,y \in X, x \ne y$ and for $\alpha,\beta,\gamma,\kappa,\delta,\eta,\mu \in [0,1)$ such that $2\alpha+2\kappa+4\delta+4\eta+2\mu<2$ then T has a unique fixed point in X.

International Journal of Mathematics and Mathematical Sciences
ISSN: 2454-7271, Copyright@universalprint
Volume: 01 No.: 02, Pages: 59-69 , Published:Dec.,2015
Web: www.universalprint.org , Publication: Universal Print
Title Key: Some Fixed Point Results for Single and Two Maps…

Proof. Define $Tx_n = x_{n+1}$ then

$$
\begin{aligned}
& d(x_{n+1}, x_n) \\
& = d(Tx_n, Tx_{n-1}) \le \alpha \frac{d(x_n, Tx_n) d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_{n-1}) d(x_{n-1}, Tx_n)}{d(x_n, x_{n-1})} \\
& + \beta \frac{d(x_n, Tx_{n-1})[d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})]}{d(x_n, x_{n-1}) + d(x_{n-1}, Tx_{n-1}) + d(x_{n-1}, Tx_n)} \\
& + \gamma \frac{d(x_n, Tx_n) d(x_{n-1}, Tx_n) + d(x_{n-1}, Tx_{n-1}) d(x_n, Tx_{n-1})}{d(x_n, Tx_n) + d(x_{n-1}, Tx_n) + d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_{n-1})} \\
& + \kappa \frac{d(x_n, Tx_n) d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_{n-1}) d(x_{n-1}, Tx_n)}{d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)} \\
& + \delta[d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})] + \eta[d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})] + \mu d(x_n, x_{n-1}) \\
& \le (\alpha + \frac{\gamma}{2} + \delta + \eta) d(x_n, x_{n+1}) + (\kappa + \delta + \eta + \mu) d(x_n, x_{n-1})
\end{aligned}
$$

$$\therefore d(x_{n+1}, x_n) \le \frac{(\kappa + \delta + \eta + \mu)}{1 - (\alpha + \frac{\gamma}{2} + \delta + \eta)} d(x_n, x_{n-1})$$

hence,

$$d(x_{n+1}, x_n) \le \lambda d(x_n, x_{n-1}) \qquad \textit{where} \quad \lambda = \frac{(\kappa + \delta + \eta + \mu)}{1 - (\alpha + \frac{\gamma}{2} + \delta + \eta)}, \quad 0 \le \lambda < 1.$$

continuing the same process we get

$$d(x_{n+1}, x_n) \le \lambda^n d(x_1, x_0)$$

Now for any $m, n \quad m > n$ using triangle inequality we have

$$
\begin{aligned}
& d(x_n, x_m) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \ldots + d(x_{m-1}, x_m) \\
& \le \lambda^n d(x_1, x_0) + \lambda^{n+1} d(x_1, x_0) + \lambda^{n+2} d(x_1, x_0) + \ldots + \lambda^{m-1} d(x_1, x_0) \\
& \le (\lambda^n + \lambda^{n+1} + \lambda^{n+2} + \ldots + \lambda^{m-1} d(x_1, x_0) = \frac{\lambda^n}{1 - \lambda} d(x_1, x_0)
\end{aligned}
$$

For any $\varepsilon > 0$ choose a positive number $N \ge 0$ such that $\frac{\lambda^N}{1-\lambda} d(x_1, x_0) < \varepsilon$

Then for any $m > n \ge N, \quad d(x_n, x_m) \le \frac{\lambda^n}{1-\lambda} d(x_1, x_0) \le \frac{\lambda^N}{1-\lambda} d(x_1, x_0) < \varepsilon$

International Journal of Mathematics and Mathematical Sciences
ISSN: 2454-7271, Copyright@universalprint
Volume: 01 No.: 02, Pages: 59-69 , Published:Dec.,2015
Web: www.universalprint.org , Publication: Universal Print
Title Key: Some Fixed Point Results for Single and Two Maps…

Hence $\{x_n\}$ is a Cauchy sequence in X . Since X is complete so there exists a point $u \in X$ such that $x_n \to u$ as $n \to \infty$. Further continuity of T in X implies

$$T(u) = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = u$$

therefore u is the fixed point of T .

Uniqueness:

If possible, let u and v are two fixed point of T so that by definition we have $Tu = u$ & $Tv = v$. So

$$d(u,v) = d(Tu,Tv)$$

$$\leq \alpha \frac{d(u,Tv)d(v,Tv) + d(u,Tv)d(v,Tu)}{d(u,v)} + \beta \frac{d(u,Tv)[d(u,Tu) + d(v,Tv)]}{d(u,v) + d(v,Tv) + d(v,Tu)}$$

$$+\gamma \frac{d(u,Tu)d(v,Tu) + d(v,Tv)d(u,Tv)}{d(u,Tu) + d(v,Tu) + d(v,Tv) + d(u,Tv)} + \kappa \frac{d(u,Tu)d(u,Tv) + d(v,Tv)d(v,Tu)}{d(u,Tv) + d(v,Tu)}$$

$$+\delta[d(u,Tu) + d(v,Tv)] + \eta[d(v,Tu) + d(u,Tv)] + \mu d(u,v)$$

which implies $d(u,v) \leq (\alpha + 2\eta + \mu) d(u,v)$

which is a contradiction, since $2\alpha + 2\kappa + 4\delta + 4\eta + 2\mu < 2$.

Hence $d(u,v) = 0 \Rightarrow u = v$. This completes the proof of the theorem.

Remark:

In theorem (2) If

1. $\alpha = \beta = \gamma = \kappa = \delta = \eta = 0$ then the theorem is reduced to Banach [11]
2. $\alpha = \beta = \gamma = \kappa = \eta = \mu = 0$ then the theorem is reduced to Kannan [8]
3. $\alpha = \beta = \gamma = \kappa = \eta = 0$ then the theorem is reduced to Chatterjee [10]
4. $\alpha = \beta = \gamma = \kappa = \delta = 0$ then the theorem is reduced to Fisher [1]
5. $\alpha = \beta = \gamma = \kappa = 0$ then the theorem is reduced to Riech [12]
6. $\alpha = \beta = \gamma = \delta = \eta = \mu = 0$ then theorem is reduced to M. S. Khan [6]
7. $\kappa = 0$ then the theorem is reduced to R. Bhardwas et.al [7]

Now we establish a result for which T is not necessarily continuous in X but T^r is continuous for some positive integer r then T has a unique fixed point in X .

Theorem 3 *Let T be a self mapping defined on a complete metric space* (X,d) *such that the condition (1) holds. If for some positive integer r,* T^r *is continuous then T has a unique fixed point in X.*

Proof. Let us define a sequence $\{x_n\}$ as in theorem (2), then clearly it converges to some point u of X. So we can define a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which also converges to the same point u of X . Now

$$T^r u = T^r(\lim_{k\to\infty} x_{n_k}) = \lim_{k\to\infty}(T^r x_{n_k}) = \lim_{k\to\infty}(x_{n_{k+1}}) = u.$$

Hence u is a fixed point of T^r .

Now we show that $Tu = u.$

Let p be the smallest positive integer such that $T^p u = u$ but $T^q \neq u, \quad for \quad q = 1,2,3...p-1$. If $p-1>0$ then

$$white.d(Tu,u) = d(Tu,T^p u)$$
$$= d(Tu,T(T^{p-1}u))$$
$$\leq \alpha \frac{d(u,Tu)d(T^{p-1}u,T^p u) + d(u,T^p u)d(T^{p-1}u,T^p u)}{d(u,T^{p-1}u)}$$
$$+\beta \frac{d(u,T^p u)[d(Tu,T^p u) + d(T^{p-1}u,T^p u)]}{d(u,T^{p-1}u) + d(T^{p-1}u,T^p u) + d(T^{p-1}u,Tu)}$$
$$+\gamma \frac{d(u,Tu)d(T^{p-1}u,Tu) + d(T^{p-1}u,T^p u)d(Tu,T^p u)}{d(u,Tu) + d(T^{p-1}u,Tu) + d(T^{p-1}u,T^p u) + d(Tu,T^p u)}$$
$$+\kappa \frac{d(u,Tu)d(u,T^p u) + d(T^{p-1}u,T^p u)d(T^{p-1}u,Tu)}{d(u,T^p u) + d(T^{p-1}u,Tu)}$$
$$+\delta[d(u,Tu) + d(T^{p-1}u,T^p u)] + \eta[d(T^{p-1}u,Tu) + d(u,T^p u)] + \mu d(u,T^{p-1}u)$$

i.e

$$d(u,Tu) \leq \frac{(\delta+\kappa+\eta+\mu)}{1-(\alpha+\frac{\gamma}{2}+\delta+\eta)} d(u,T^{p-1}u)$$

International Journal of Mathematics and Mathematical Sciences
ISSN: 2454-7271, Copyright@universalprint
Volume: 01 No.: 02, Pages: 59-69 , Published:Dec.,2015
Web: www.universalprint.org , Publication: Universal Print
Title Key: Some Fixed Point Results for Single and Two Maps…

$$\therefore \quad d(u,Tu) \le \lambda d(u,T^{p-1}u) \le \ldots \le \lambda^p d(u,Tu)$$

$$where \qquad \lambda = \frac{(\delta+\kappa+\eta+\mu)}{1-(\alpha+\frac{\gamma}{2}+\delta+\eta)} < 1$$

a contradiction, hence $d(u,Tu) = 0 \Rightarrow \quad u = Tu$

The uniqueness can be followed as in theorem (2). This completes the proof of the theorem.

Theorem 4 *Let S and T be mappings of a complete metric space (X, d) into itself. Suppose that there exists a non negative real number* α *and* β *such that* $\alpha + 2\beta < 1$ *and*

$$d(Tx,Sy) \le \alpha \frac{d(x,Tx)d(x,Sy)+d(y,Sy)d(y,Tx)}{d(x,Sy)+d(y,Tx)}$$
$$+\beta max\{d(x,Tx)+d(y,Sy), d(y,Sy)+d(x,y), d(x,Tx)+d(x,y)\}$$

for all $x, y \in X$ then S and T have a unique common fixed point.

Proof. Let $x_0 \in X$. Define the sequence $\{x_n\}$ by
$x_{2n+1} = S(x_{2n})$, $x_{2n+2} = T(x_{2n+1})$, n = 0, 1, 2, ... then we have

$$d(x_1,x_2) = d(Sx_0,Tx_1) = d(Tx_1,Sx_0)$$
$$\le \alpha \frac{d(x_1,Tx_1)d(x_1,Sx_0)+d(x_0,Sx_0)d(x_0,Tx_1)}{d(x_1,Sx_0)d(x_0,Tx_1)}$$
$$+\beta max\{d(x_1,Tx_1)+d(x_0,Sx_0), d(x_0,Sx_0)+d(x_1,x_0), d(x_1,Tx_1)+d(x_1,x_0)\}$$
$$= \alpha \frac{d(x_1,x_2)d(x_1,x_1)+d(x_0,x_1)d(x_0,x_2)}{d(x_1,x_1)+d(x_0,x_2)}$$
$$+\beta max\{d(x_1,x_2)+d(x_0,x_1), d(x_0,x_1)+d(x_1,x_0), d(x_1,x_2)+d(x_1,x_0),\}$$
$$= \alpha d(x_0,x_1) + \beta\{d(x_1,x_2)+d(x_0,x_1)\}$$

$$(1-\beta)d(x_1,x_2) \le (\alpha+\beta)d(x_0,x_1)$$

$$d(x_1,x_2) \le \frac{\alpha+\beta}{1-\beta} d(x_0,x_1)$$

International Journal of Mathematics and Mathematical Sciences
ISSN: 2454-7271, Copyright@universalprint
Volume: 01 No.: 02, Pages: 59-69 , Published:Dec.,2015
Web: www.universalprint.org , Publication: Universal Print
Title Key: Some Fixed Point Results for Single and Two Maps…

Put $\lambda = \dfrac{\alpha+\beta}{1-\beta}$ where $0 \le \lambda < 1$

Then $d(x_1, x_2) \le \lambda d(x_0, x_1)$

Similarly we can show, $d(x_2, x_3) \le \lambda d(x_1, x_2)$

In general we have $d(x_n, x_{n+1}) \le \lambda^n d(x_0, x_1)$

Hence $\{x_n\}$ ia Cauchy sequence. Since X is a complete metric space, so the sequence $\{x_n\}$ converges to some point x in X.

For the point x,

$$
\begin{aligned}
d(x,Tx) &\le d(x,x_{n+1}) + d(Tx_n, Tx) \\
&= d(x,x_{n+1}) + \alpha \frac{d(x_n,Tx_n)d(x_n,Tx) + d(x,Tx)d(x,Tx_n)}{d(x_n,Tx) + d(x,Tx_n)} \\
&\quad + \beta max\{d(x_n,Tx_n) + d(x,Tx), d(x,Tx) + d(x_n,x), d(x_n,Tx_n) + d(x_n,x)\} \\
&= d(x,x_{n+1}) + \alpha \frac{d(x_n,x_{n+1})d(x_n,Tx) + d(x,Tx)d(x,x_{n+1})}{d(x_n,Tx) + d(x,x_{n+1})} \\
&\quad + \beta max\{d(x_n,x_{n+1}) + d(x,Tx), d(x,Tx) + d(x_n,x), d(x_n,x_{n+1}) + d(x_n,x)\}
\end{aligned}
$$

Taking limit as $n \to \infty$ we have,

$$d(x,Tx) \le \beta d(x,Tx) \text{ a contradiction.}$$

$\therefore d(x,Tx) = 0 \Rightarrow x = Tx$.

Hence x is the fixed point of T. Similarly following the same process we can show that x is the fixed point of S . Hence x is the common fixed point of T and S.

Uniqueness:

To show x is a unique common fixed point of the mappings T ans S if possible let y be a fixed point of S.

$$
\begin{aligned}
d(x,y) &= d(Tx,Sy) \\
&\le \alpha \frac{d(x,Tx)d(x,Sy) + d(y,Sy)d(y,Tx)}{d(x,Sy) + d(y,Tx)} \\
&\quad + \beta max\{d(x,Tx) + d(y,Sy), d(y,Sy) + d(x,y), d(x,Tx) + d(x,y)\} \\
&= \alpha \frac{d(x,x)d(x,y) + d(y,y)d(y,x)}{d(x,y) + d(y,x)} \\
&\quad + \beta max\{d(x,x) + d(y,y), d(y,y) + d(x,y), d(x,x) + d(x,y)\}
\end{aligned}
$$

International Journal of Mathematics and Mathematical Sciences
ISSN: 2454-7271, Copyright@universalprint
Volume: 01 No.: 02, Pages: 59-69 , Published:Dec.,2015
Web: www.universalprint.org , Publication: Universal Print
Title Key: Some Fixed Point Results for Single and Two Maps…

$$d(x,y) \le \beta d(x,y)$$

which is a contradiction, since $\alpha + 2\beta < 1$, Hence $d(x,y) = 0 \Rightarrow x = y$. This coplets the proof of the theorem.

Remark:

If $\beta = 0$ we get theorem 2 of M.S. Khan [6]

If $\alpha = 0$ we get theorem 3.7 of R. Shrivastva et. al. [9].

If $S = T$ then we get the following corollary

Corrollary 1

$$d(Tx,Ty) = \alpha \frac{d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx)}{d(x,Ty) + d(y,Tx)} + \beta max\{d(x,Tx) + d(y,Ty), d(y,Ty) + d(x,y), d(x,Tx) + d(x,y)\}$$

Remark:

If $\alpha = 0$ then we get A -Contraction introduced by M. Akram et.al [5].

If $\beta = 0$ we get theorem 1 of M. S. Khan [6]

Again the result of theorem (2) can be further generalized. In this case, the mapping T is neither continuous nor satisfies the condition (1) but T^m for some positive integer m satisfies the same rational condition and contiuous, T still consumes a unique fixed point in X.

Theorem 5 *Let T be continuous self map defined in a complete metric space (X, d) such that for some positive integer m, satisfies the condition*

$$d(T^m x, T^m y) \le \alpha \frac{d(x,T^m x)d(y,T^m y) + d(x,T^m y)d(y,T^m x)}{d(x,y)} + \beta \frac{d(x,T^m y)[d(x,T^m x) + d(y,T^m y)]}{d(x,y) + d(y,T^m y) + d(y,T^m x)} + \gamma \frac{d(x,T^m x)d(y,T^m x) + d(y,T^m y)d(x,T^m y)}{d(x,T^m x) + d(y,T^m x) + d(y,T^m y) + d(x,T^m y)} + \kappa \frac{d(x,T^m x)d(x,T^m y) + d(y,T^m y)d(y,T^m x)}{d(x,T^m y) + d(y,T^m x)}$$

International Journal of Mathematics and Mathematical Sciences
ISSN: 2454-7271, Copyright@universalprint
Volume: 01 No.: 02, Pages: 59-69 , Published:Dec.,2015
Web: www.universalprint.org , Publication: Universal Print
Title Key: Some Fixed Point Results for Single and Two Maps…

$$+\delta[d(x,T^m x)+d(y,T^m y)]+\eta[d(y,T^m x)+d(x,T^m y]+\mu d(x,y)$$

for all $x,y\in X, x\neq y$ and for $\alpha,\beta,\gamma,\kappa,\delta,\eta,\mu\in[0,1)$ such that $2\alpha+2\kappa+4\delta+4\eta+2\mu<2$, if T^m is continuous then T has a fixed point in X.

Proof. By theorem (3), it is obvious that T^m has a unique fixed point u in X i.e $T^m(u)=u$. Also

$$T(u)=T(T^m u)=T^m(Tu)$$

From both relations we conclude that $T(u)=u.$ i.e T has a fixed point u in X. This completes the proof of theorem.

Theorem 6 *Let $\{T_n\}$ be a sequence of mappings of a complete metric space (X, d) into itself. Let x_n be a fixed point of $\{T_n\}$ (n = 1, 2, ...) and suppose $\{T_n\}$ converges uniformly to T_0. If T_0 satisfies the condition*

$$d(T_0x,T_0y)\leq\alpha\frac{d(x,T_0x)d(x,T_0y)+d(y,T_0y)d(y,T_0x)}{d(x,T_0y)+d(y,T_0x)}$$

$$+\beta\frac{d(x,T_0y)[d(x,T_0x)+d(y,T_0y)]}{d(x,y)+d(y,T_0y)+d(y,T_0x)}+\gamma d(x,y)$$

for all $x,y\in X, x\neq y$ and for $\alpha,\beta,\gamma,\in[0,1)$ such that $\alpha+\beta+\gamma<1$ then $\{x_n\}$ converges to the fixed point x_0 of T_0.

Proof. From Theorem (2) and by given remarks conclude that T_0 has a unique fixed point satisfying the given rational expression. Let $\varepsilon>0$ be given, then there exists a natural number N such that

$$d(T_nx,T_0x)<\frac{\varepsilon}{1-(\alpha+\beta+\gamma)}\quad \text{for all } x\in X \text{ and } n>N.$$

$$d(x_n,x_0)=d(T_nx_n,T_0x_0)\leq d(T_nx_n,T_0x_n)+d(T_0x_n,T_0x_0)$$

$$\leq d(T_nx_n,T_0x_n)+\alpha\frac{d(x_n,T_0x_n)d(x_n,T_0x_0)+d(x_0,T_0x_0)d(x_0,T_0x_n)}{d(x_n,T_0x_0)+d(x_0,T_0x_n)}$$

$$+\beta\frac{d(x_n,T_0x_0)[d(x_n,T_0x_n)+d(x_0,T_0x_0)]}{d(x_n,x_0)+d(x_0,T_0x_0)+d(x_0,T_0x_n)}+\gamma d(x_n,x_0)$$

$$=d(T_nx_n,T_0x_n)+\alpha\frac{d(x_n,T_0x_n)d(x_n,x_0)+d(x_0,x_0)d(x_0,T_0x_n)}{d(x_n,x_0)+d(x_0,T_0x_n)}$$

$$+\beta\frac{d(x_n,x_0)[d(x_n,T_0x_n)+d(x_0,x_0)]}{d(x_n,x_0)+d(x_0,x_0)+d(x_0,T_0x_n)}+\gamma d(x_n,x_0)$$
$$\leq d(T_nx_n,T_0x_n)+(\alpha+\beta+\gamma)d(x_n,x_0)$$

i.e.

$$d(x_n,x_0)\leq\frac{1}{1-(\alpha+\beta+\gamma)}d(T_nx_n,T_0x_n)<\varepsilon \text{ for } n>N.$$

This shows that $\{x_n\}$ converges to x_0. This completes the proof of the theorem.

Remark:
In the above theorem, if $\beta=\gamma=0$ then we get theorem 3 of M. S. Khan[6].

References

[1] B. Fisher, *A fixed point theorem for compact metic space,* Publ. Inst. Math, **25**(1976), 192-194.

[2] D. Panthi and K. Jha, *A Common Fixed Point of Weakly Compatible Mappings in Dislocated Metric Space,* Kath. Univ. J. Sci. Engg. Tech, **8** (2012), 25- 30.

[3] D. S. Jaggi, *Some unique fixed point theorems,* Indian J. Pure Appl. Math. **8**(2) (1977), 223 - 230.

[4] D. S. Jaggi and Dass B. K., *An extension of Banach contraction theorem through rational expression,* Bull. Call. Math.,(1980), 261- 266.

[5] M. Akram, A. A.Zafar and A. A. Siddiqui, *A general class of contractions: A-Contraction,* Novi Sad J. math, **38**(1), (2008), 25-33.

[6] M. S. Khan, *A fixed Point theorem for metric spaces,* Riv. Mat. Univ. Parma, **4**(1977), 53 - 57.

[7] R. Bhardwas, S. S. Rajput and R. N. Yadav, *Application of Fixed Point theory in Metric Space*, Thai J. Math. **5**(2),(2007), 253- 259.

[8] R. Kannan, *Some results on fixed points,* Bull. Cal. Math. Soc., **60**(1968), 71-76.

[9] R. Shrivastava, Z. K. Ansari and M. Sharma, *Some Results on Fixed Points in Dislocated and Dislocated Quasi-Metric Spaces,* J. Advance Studies in Topology, **3** (1)(2012), 25-31.

[10] S. K. Chaterjee,*Fixed Point Theorems,* Compts. Rend. Acad. Bulgeria Sci., **25**(1972), 727-730.

[11] S. Banach, *Sur les operations dans les ensembles abstraits et leur applications aux equations integrales,* fundamental Mathematicae, **3**(7),(1922), 133-181.

[12] S. Reich, *Some remarks concerning cotraction mappings*, Canada Math. Bull., **1**(1971), 121 - 124.

[13] S.S. Bellale and S.B. Birajdar, *On Existence Theorem for Nonlinear Functional Two Point Boundary Value Problems in Banach Algebras,* Int. Jr. Universal Mathematics and Mathematical Sciences 1(1) (2015), 47-58.

[14] Bapurao C. Dhage, S.S. Bellale, *Existence Theorem for Perturbed Abstract Measure Differential Equations,* Nonlinear Analysis 71(2009) e319-e328, doi:10.1016/ j.na2008. 11.057.