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Fixed Point Results via Implicit Type Cyclic Contractive Conditions in Complete Metric Spaces and Application

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Abstract

We introduce an implicit type cyclic contractive mapping for a map in metric space and develop existence and uniqueness results of fixed points for such mappings. Examples are given to support the usability of our results. Some consequences are given after the main results. Fixed point results for well posed property, limit shadowing property and two metric spaces are also given. Finally, an application to the study of existence and uniqueness of solutions for a class of nonlinear integral equations is given. Our results extend, unify and generalize several well known comparable results in the literature.

Keywords: Fixed point, Implicit contraction; fixed point.

Subject Classification: Primary 54H25, Secondary 47H10.

1. Introduction

Metric fixed point theory has primary applications in functional analysis. Extension of fixed point theory to generalized structures as cone metrics, partial metric spaces and quasi-metric spaces has received a lot of attention (see, for instance, [1 - 18] and references mentioned therein). The study of fixed points of mappings satisfying certain contractive conditions has been at the center of vigorous research activity.

Celebrated Banach's [2] contraction mapping principle is one of the cornerstone in the development of nonlinear analysis. Recall that a self-mapping $T: X \to X$, where (X, d) is a metric space, is said to be a contraction if there exists $0 < k < 1$ such that for all $x, y \in X$, (1.1) $$d(Tx,Ty) \leq kd(x,y)$$

Inequality (1) implies continuity of T. A natural question is that whether we can find contractive conditions which will imply existence of fixed point in a complete metric space but will not imply continuity.

One of the remarkable generalizations of the Banach contraction principle was reported by Kirk, Srinivasan and Veeramani [11] via cyclic contraction. A mapping $T: A \cup B \to A \cup B$ is called cyclic if $T(A) \subseteq B$ and $T(B) \subseteq A$, where A, B are nonempty subsets of a metric space (X,d). Moreover, T is called cyclic contraction if there exists $k \in (0,1)$ such that $d(Tx,Ty) \leq kd(x,y)$ for all $x \in A$ and $y \in B$. Notice that although a contraction is continuous, cyclic contraction need not to be. This is one of the important gains of this theorem.

Definition 1.1 (see [11,13]) Let (X,d) be a metric space. Let p be a positive integer, and $A_1, A_2, \ldots\ldots A_p$ be nonempty subsets of X. Then $Y = \cup_{i=1}^{p} A_i$ is said to be a cyclic representation of Y with respect to T if

(i). $A_i, i = 1,2,\ldots, p$, are nonempty closed sets, and

(ii). $T(A_1) \subseteq A_2, \ldots, T(A_{p-1}) \subseteq A_p, T(A_p) \subseteq A_1$.

Following the paper in [11], a number of fixed point theorems on cyclic representation of Y with respect to a self-mapping T have appeared (see e.g. [1, 9, 10, 13, 14, 17]).

In this work, we introduce a class of implicit type cyclic contractive of mappings and investigate the existence and uniqueness of fixed points for such mappings. Our main result generalizes and improves many existing theorems in the literature. Some consequences are given after the main results. Moreover, we state some examples and present an application to analyze the existence and uniqueness of solutions for a class of nonlinear integral equations.

2. Main Results

To complete the results, we need following setting of implicit contraction.

We consider the set Φ of functions $\varphi : \mathbb{R}^{+5} \to \mathbb{R}^{+}$ satisfying the following properties:

(i). φ is continuous;

(ii). φ is nondecreasing with respect to $\leq$ in the 4th and 5th variable;

(iii). There are $h_1 > 0$ and $h_2 > 0$ such that $h = h_1 h_2 < 1$ and if $u, v \in \mathbb{R}^{+}$ satisfy $u \leq \varphi(v, v, u, u+v, 0)$, then $u \leq h_1 v$ and if $u, v \in \mathbb{R}^{+}$ satisfy $u \leq \varphi(v, u, v, 0, u+v)$, then $u \leq h_2 v$;

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(iv). If $u \in \mathbb{R}^+$ is such that $u \le \varphi(u,0,0,u,u)$ or $u \le \varphi(0,u,0,0,u)$ or $u \le \varphi(0,0,u,u,0)$, then $u = 0$.

Example 2.1 Let $\varphi : \mathbb{R}^{+5} \to \mathbb{R}^+$ denote

(2) $$\varphi(t_1, t_2, t_3, t_4, t_5) = k \max\left\{t_1, t_2, t_3, \frac{1}{2}(t_4 + t_5)\right\}, \text{ for } k \in (0,1).$$

Then, $\varphi \in \Phi$.

We state the notion of implicit type cyclic contractive mapping as follows.

Definition 2.1 Let (X,d) be a metric space. Let p be a positive integer, and $A_1, A_2, \ldots \ldots A_p$ be nonempty subsets of X . and $Y = \cup_{i=1}^{p} A_i$ an operator $T: Y \to Y$ satisfies a implicit type cyclic contractive mapping for some $\varphi \in \Phi$. if

(i). $Y = \cup_{i=1}^{p} A_i$ is said to be a cyclic representation of Y with respect to T;

(ii). for any $(x,y) \in A_i \times A_{i+1}$, i =1 , 2 ,...,p (where $A_{p+1} = A_1$),

(3) $$d(Tx,Ty) \le \varphi\big(d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big).$$

It is easy to acquire the following examples of implicit type cyclic contractive mapping from

Example 2.2. Let X = [0,1] with the usual metric. Suppose $A_1 = \left[0,\frac{1}{2}\right]$ and $A_2 = [\frac{1}{2}, 1]$ and $X = \cup_{i=1}^{2} A_i$. Define $T: X \to X$ such that

$$Tx = \begin{cases} \frac{1}{2}, & x \in [0,1) \\ 0, & x = 1 \end{cases}.$$

Clearly, A_1 and A_2 are closed subsets of X. Moreover $T(A_i) \subset A_{i+1}$ for $i = 1,2$, so that $\cup_{i=1}^{2} A_i$, is a cyclic representation of X with respect to T. Furthermore, if $\varphi : \mathbb{R}^{+5} \to \mathbb{R}^+$ denote

(5) $$\varphi(t_1, t_2, t_3, t_4, t_5) = \frac{3}{4} \max\left\{t_1, t_2, t_3, \frac{1}{2}(t_4 + t_5)\right\},$$

where $t_1 = d(x,y), t_2 = d(x,Tx), t_3 = d(y,Ty), t_4 = d(x,Ty)$ and $t_5 = d(y,Tx)$, for all x, y $\in$ X. Then, $\varphi \in \Phi$.

Next we show that T satisfies implicit type cyclic contractive condition. We shall distinguish the following cases:

(1) For $x \in A_1, y \in A_2$,

- When $x \in \left[0,\frac{1}{2}\right]$ and $y \in [\frac{1}{2}, 1)$, we deduce $d(Tx,Ty) = 0$ and equation (3) is trivially satisfied.
- When $x \in \left[0,\frac{1}{2}\right]$ and $y = 1$, we deduce $d(Tx,Ty) = \frac{1}{2}$ and

$t_1 = |x-1|, t_2 = \left|x - \frac{1}{2}\right|, t_3 = 1, t_4 = x, t_3 = \frac{1}{2},$ then $\varphi(t_1, t_2, t_3, t_4, t_5) = \frac{3}{4}$,

Equation (3) holds as it reduces to $\frac{1}{2} \le \frac{3}{4}$.

(2) For $y \in A_1, x \in A_2$,

- When $x \in \left[\frac{1}{2}, 1\right)$ and $y \in \left[0, \frac{1}{2}\right]$ we deduce $d(Tx, Ty) = 0$ and equation (3) is trivially satisfied.
- When $x \in 0$ and $y \in \left[0, \frac{1}{2}\right]$ we deduce $d(Tx, Ty) = \frac{1}{2}$ and

$t_1 = |1-y|, t_2 = 1, t_3 = \left|y - \frac{1}{2}\right|, t_4 = \frac{1}{2}, t_3 = y,$then $\varphi(t_1, t_2, t_3, t_4, t_5) = \frac{3}{4}$, equation (3) holds as it reduces to $\frac{1}{2} \le \frac{3}{4}$.

Hence T is a implicit type cyclic contractive mapping.

Our main result is the following.

Theorem 2.1 Let (X, d) be a complete metric space, Let $p \in \mathbb{N}$, $A_1, A_2, \ldots\ldots A_p$ be nonempty subsets of X . and $Y = \cup_{i=1}^{p} A_i$ Suppose $T: Y \to Y$ is a implicit type cyclic contractive mapping, for some $\varphi \in \Phi$. Then T has a unique fixed point. Moreover, the fixed point of T belong to $\cap_{i=1}^{p} A_i$.

Proof. Let $x_0 \in A_1$ (such a point exists since $A_1 \neq \emptyset$). Define the sequence $\{x_n\}$ in X by:

$$x_{n+1} = Tx_n, \quad n = 0,1,2, \ldots .$$

We shall prove that

(6) $$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

We suppose that $d(x_n, x_{n+1}) > 0$ for all n. Then, from the condition (II), we have

$d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1})$
$\le \varphi\big(d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), d(x_n, Tx_{n+1}), d(x_{n+1}, Tx_n)\big)$
$= \varphi\big(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+2}), d(x_{n+1}, x_{n+1})\big)$
$\le \varphi(0,0, d(x_{n+1}, x_{n+2}), d(x_n, x_{n+2}), 0)$.

From (iv) this implies that $d(x_{n+1}, x_{n+2}) = 0$, that is,

$$x_{n+1} = x_{n+2}.$$

Following the similar arguments, we obtain $x_{n+2} = x_{n+3}$ and so on. Thus $\{x_n\}$ becomes a constant sequence and x_n is the fixed point of T.

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Take $d(x_n, x_{n+1}) > 0$ for each n. From the condition (I), we observe that for all n, there exists $i = i(n) \in \{1,2,\dots,p\}$ such that $(x_n, x_{n+1}) \in A_i \times A_{i+1}$. From (II), we have

$d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1})$
$\leq \varphi(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+2}), 0)$
$= \varphi(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}), 0)$.

By (iii), we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \varphi\big(d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), 0, d(x_{n+1}, x_{n-1})\big) \\ &= \varphi\big(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), 0, d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})\big). \end{aligned}$$

By (iii), we obtain

(8) $$d(x_{n+1}, x_n) = h_2 d(x_n, x_{n-1}).$$

Combining (7) and (8), we have

$$d(x_{n+1}, x_{n+2}) = hd(x_n, x_{n-1}).$$

Continuing this process, we get

(9) $$d(x_{n+1}, x_{n+2}) = h^n d(x_1, x_2).$$

Again, using (II), we have

$$\begin{aligned} d(x_{n+3}, x_{n+2}) &= d(Tx_{n+2}, Tx_{n+1}) \\ &\leq \varphi\big(d(x_{n+2}, x_{n+1}), d(x_{n+2}, x_{n+3}), d(x_{n+1}, x_{n+2}), 0, d(x_{n+3}, x_{n+1})\big) \\ &\leq \varphi\big(d(x_{n+2}, x_{n+1}), d(x_{n+2}, x_{n+3}), d(x_{n+1}, x_{n+2}), 0, d(x_{n+3}, x_{n+2}) + d(x_{n+2}, x_{n+1})\big). \end{aligned}$$

From (iii), we get

$$d(x_{n+2}, x_{n+3}) = h_2 d(x_{n+1}, x_{n+2}).$$

Using (9), we obtain

$$d(x_{n+2}, x_{n+3}) = h_2 h^n d(x_{n+1}, x_{n+2}).$$

From (9) and (10), we get

(11) $d(x_n, x_{n+1}) \leq \frac{max\{1,h_2\}}{\sqrt{h}} \left(\sqrt{h}\right)^n d(x_1, x_2)$. for all $n = 2,3,\dots$,

From (11) and using the triangular inequality, for all $n, m \in \mathbb{N}$ with $m > n$, we have

$$d(x_n, x_{n+m}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m)$$
$$\leq \frac{max\{1,h_2\}}{\sqrt{h}} \left(\left(\sqrt{h}\right)^n + \left(\sqrt{h}\right)^{n+1} + \cdots + \left(\sqrt{h}\right)^{m-1}\right) d(x_1, x_2)$$
$$\leq \frac{max\{1,h_2\}}{\sqrt{h}} \frac{\left(\sqrt{h}\right)^n}{1-\sqrt{h}} d(x_1, x_2).$$

Since $0 < h < 1$, there exists $N \in \mathbb{N}$, such that

(12) $$\frac{max\{1,h_2\}}{\sqrt{h}} \frac{\left(\sqrt{h}\right)^n}{1-\sqrt{h}} d(x_1, x_2) \to 0, \text{ for all } n > N.$$

Thus, for all $n, m \in \mathbb{N}$,

$$d(x_n, x_m) \leq \frac{max\{1, h_2\}}{\sqrt{h}} \frac{(\sqrt{h})^n}{1-\sqrt{h}} d(x_1, x_2) \to 0,$$

and so the sequence $\{x_n\}$ is a Cauchy sequence in X.
Since (X, d) is complete, there exists $x^* \in X$ such that

(13) $$\lim_{n\to\infty} x_n = x^*.$$

We shall prove that

(14) $$x^* \in \cap_{i=1}^{p} A_i.$$

From condition (I), and since $x_0 \in A_1$, we have $\{x_{np}\}_{n\geq 0} \subseteq A_1$. Since A_1 is closed, from (13), we get that $x^* \in A_1$. Again, from the condition (I), we have $\{x_{np+1}\}_{n\geq 0} \subseteq A_2$. Since A_2 is closed, from (13), we get that $x^* \in A_2$. Continuing this process, we obtain (14).
Now, we shall prove that x^* is a fixed point of T. Indeed, from (14), since for all , there exists $i(n) \in \{1,2, \dots, p\}$ such that $x_n \in A_{i(n)}$, Appling (II) with $x = x^*$ and $y = x_n$, we obtain for all n

$$d(Tx^*, x_{n+1}) = d(Tx^*, Tx_n)$$
$$\leq \varphi\big(d(x^*, x_n), d(x^*, Tx^*), d(x_n, Tx_n), d(x^*, Tx_n), d(x_n, Tx^*)\big)$$

(15) $$= \varphi\big(d(x^*, x_n), d(x^*, Tx^*), d(x_n, x_{n+1}), d(x^*, x_{n+1}), d(x_n, Tx^*)\big).$$

Passing to the limit $n \to \infty$ in (15), using (13), we get

$$d(x^*, Tx^*) \leq \varphi\big(d(x^*, x^*), d(x^*, Tx^*), d(x^*, x^*), d(x^*, x^*), d(Tx^*, x^*)\big)$$
$$= \varphi\big(0, d(x^*, Tx^*), 0,0, d(Tx^*, x^*)\big).$$

From (iv), this implies that $d(x^*, Tx^*) = 0$, that is,

(16) $$Tx^* = x^*,$$

That is, x^* is a fixed point of T.
Finally, we prove that x^* is the unique fixed point of T. Assume that y^* is another fixed point of T, that is $Ty^* = y^*$. From the condition (I), this implies that $y^* = \cap_{i=1}^{p} A_i$. Then we can apply (II) for $x = x^*$ and $y = y^*$. We obtain

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq \varphi\big(d(x^*, y^*), d(x^*, Tx^*), d(y^*, Ty^*), d(x^*, Ty^*), d(Tx^*, y^*)\big).$$

Since x^* and $y^*$are fixed points of T, we can show easily that $d(x^*, y^*) = 0$. If $d(x^*, y^*) > 0$, we get

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq \varphi\big(d(x^*, y^*), d(x^*, Tx^*), d(y^*, Ty^*), d(x^*, Ty^*), d(Tx^*, y^*)\big)$$
$$= \varphi\big(d(x^*, y^*), 0,0, d(x^*, y^*), d(x^*, y^*)\big).$$

From (iv), we get $d(x^*, y^*) = 0$, that is $x^* = y^*$. Thus we proved the uniqueness of the fixed point.

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Remark 2.1 Theorem 2.1 is generalized form to Theorem 2.1 in [13].

If in Theorem 2.1, we take $A_i = X$ for $i = 1,2,\dots,m,$ we obtain the following result.

Corollary 2.1 Let (X,d) be a complete metric space and $T\colon X \to X$ a mapping such that for any $x,y \in X$

$$d(Tx,Ty) \le \varphi\big(d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big).$$

where $\varphi \in \Phi$. Then T has a unique fixed point $z \in X$

Remark 2.2. Corollary 2.1 extends and generalizes many existing fixed point theorems in the literature [2, 3, 4, 5, 7, 8, 15, 16].
Using the obtained result given by Theorem 2.1, we will prove the following theorem.

Theorem 2.2. Let (X,d) be a metric space. Let p be a positive integer, and $A_1, A_2, \dots \dots . A_p$ A_p be nonempty subsets of X and $Y = \cup_{i=1}^{p} A_i$ an operator $T\colon Y \to Y$. Suppose that there exists three positive constants A,B,C with A+2B+2C<1 such that

(i). $Y = \cup_{i=1}^{p} A_i$ is said to be a cyclic representation of Y with respect to T;

(ii). for any $(x,y) \in A_i \times A_{i+1}$, i =1 , 2 ,...,p (where $A_{p+1} = A_1$),

(17)

$$d(Tx,Ty) \le Ad(x,y) + B\,[d(x,Tx) + d(y,Ty)] + C[d(x,Ty) + d(Tx,y)],$$

Then T has a unique fixed point. Moreover, the fixed point of T belong to $\cap_{i=1}^{p} A_i$.

Proof. Define $\varphi\colon \mathbb{R}^{+^5} \to \mathbb{R}^+$ by

$$\varphi(u_1,u_2,u_3,u_4,u_5) = Au_1 + B(u_2 + u_3) + C(u_4 + u_5), \quad \text{for all } u_i \in \mathbb{R}^+.$$

Denote

$$h_1 = h_2 = \frac{A+B+C}{1-(B+C)}\ .$$

Since A+2B+2C < 1, we have $h_1 > 0$, $h_2 > 0$. If $u \le \varphi(v,v,u,u+v,0)$, we have

$$u \le Av + Bv + Bu + Cu + Cv\,,$$

which implies that $u \le h_1 v$. Now, if $u \le \varphi(v,u,v,0,u+v)$, we have

$$u \le Av + Bu + Bv + Cu + Cv\,,$$

which implies that $u \le h_2 v$. Suppose now that $u \le \varphi(u,0,0,u,u)$. We get $u \le Au + Cu$, which implies that $-[1-(A+2C)]u \in \mathbb{R}^+$. Since $A + 2C < 1$, we have $[1-(A+2C)]u \in \mathbb{R}^+$. Then $u = 0$. The same result holds if $u \le \varphi(0,u,0,0,u)$ or $u \le \varphi(0,0,u,u,0)$. Therefore, $\varphi \in \Phi$. Moreover, inequality (b) is equivalent to inequality (II). Then, to obtain the desired result, we have only to apply Theorem 2.1 for the considered function.

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Remark 2.3. Theorem 2.2 extends and generalizes the well known fixed point theorem of Hardy and Rogers [7] , [14, Theorem 7].

Remark 2.4. Theorem 2.2 extends and generalizes Theorem 1.3 in [**11**] and Theorem 3 in [**14**]. Following consequences extends and generalizes the well known fixed point theorem of Rhoades [**16**] numbered as $(21')$.

Corollary 2.2 Let (X, d) be a metric space. Let p be a positive integer, and $A_1, A_2, \ldots\ldots A_p$ be nonempty subsets of X and $Y = \cup_{i=1}^{p} A_i$ an operator $T: Y \to Y$. Suppose that there exists a constant k with $0 \le k < 1$ such that

(i). $Y = \cup_{i=1}^{p} A_i$ is said to be a cyclic representation of Y with respect to T;

(ii). for any $(x, y) \in A_i \times A_{i+1}$, i =1 , 2 ,...,p (where $A_{p+1} = A_1$),

$$d(Tx, Ty) \le k \max\left\{ d(x, y), \frac{d(x,\ Tx) + d(y,\ Ty)}{2}, \frac{d(x,\ Ty) + d(Tx, y)}{2} \right\}.$$

Then T has a unique fixed point. Moreover, the fixed point of T belongs to $\cap_{i=1}^{p} \mathcal{A}_i$

PROOF. Define $\varphi: \mathbb{R}^{+^5} \to \mathbb{R}^+$ by

$$\varphi(x_1, x_2, x_3, x_4, x_5) = k \max\left\{x_1, x_2, x_3, \frac{x_4 + x_5}{2}\right\}.$$

Since $\varphi \in \Phi$, we can apply Theorem 2.1 and obtain Corollary 2.2 .

Corollary 2.3. Let (X, d) be a complete metric space, $p \in \mathbb{N}$, $\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_p$ nonempty closed subset of X, $Y = \cup_{i=1}^{p} \mathcal{A}_i$ and $T : Y \to Y$. Suppose that there exists a constant k with $0 \le k < 1$ such that

(a) $Y = \cup_{i=1}^{p} \mathcal{A}_i$ is a cyclic representation of Y with respect to T;

(b) for any $(x, y) \in \mathcal{A}_i \times \mathcal{A}_{i+1}$, $i = 1,2, \cdots, p$ (with $\mathcal{A}_{p+1} = \mathcal{A}_1$) ,

$$d(Tx, Ty) \le k \max\left\{d(x, y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(Tx,y)}{2}\right\}.$$

Then T has a unique fixed point. Moreover, the fixed point of T belongs to $\cap_{i=1}^{p} \mathcal{A}_i$.

Proof. Define $: \mathbb{R}^{+^5} \to \mathbb{R}^+$ by

$$\varphi(x_1, x_2, x_3, x_4, x_5) = k \max\left\{x_1, \frac{x_2+x_3}{2}, \frac{x_4+x_5}{2}\right\}.$$

Since $\varphi \in \Phi$, we can apply Theorem 2.1 and obtain Corollary 2.3 .

Corollary 2.4. Let (X, d) be a complete metric space, $p \in \mathbb{N}$, $\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_p$ nonempty closed subset of X, $Y = \cup_{i=1}^{p} \mathcal{A}_i$ and $T : Y \to Y$. Suppose that there exists a constant A, B, C with $2A + 2B + C < 1$ such that

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(a) $Y = \cup_{i=1}^{p} \mathcal{A}_i$ is a cyclic representation of Y with respect to T;

(b) for any $(x, y) \in \mathcal{A}_i \times \mathcal{A}_{i+1}$, $\quad i = 1,2,\cdots,p \quad$ (with $\mathcal{A}_{p+1} = \mathcal{A}_1$),

$$d(Tx,Ty) \le A[d(x,y) + d(x,Tx)] + B[d(y,Ty) + d(x,Ty)] + Cd(Tx,y).$$

Then T has a unique fixed point. Moreover, the fixed point of T belongs to $\cap_{i=1}^{p} \mathcal{A}_i$.

Proof. Define $: \mathbb{R}^{+^5} \to \mathbb{R}^+$ by

$$\varphi(x_1, x_2, x_3, x_4, x_5) = A[x_1 + x_2] + B[x_3 + x_4] + x_5.$$

Since $\varphi \in \Phi$, we can apply Theorem 2.1 and obtain Corollary 2.4 .

Remark 2.5. Corollary 2.4 extends and generalizes the well known fixed point theorem of Hardy and Rogers [7].

We illustrate the use of Theorem 2.1 by the following :

Example 2.3. Let $= \mathbb{R}$. Mapping $d : X \to X$ is defined $d(x, y) = |x - y|$.

Clearly, the metric given above is complete metric space on X. Consider the closed subsets $\mathcal{A}_1$ and $\mathcal{A}_2$ defined by

$$\mathcal{A}_1 = \left\{\frac{-1}{2n}\right\}_{n=1}^{\infty} \cup \left\{\frac{1}{n}\right\}_{n=1}^{\infty} \cup \{0\}$$

and
$$\mathcal{A}_2 = \left\{\frac{-1}{n}\right\}_{n=1}^{\infty} \cup \left\{\frac{1}{2n-1}\right\}_{n=1}^{\infty} \cup \{0\}.$$

Define the self map T on X by

$$Tx = \begin{cases} \dfrac{-x}{x+4}, & x \in \mathcal{A}_1 \\ \dfrac{-x}{4}, & x \in \mathcal{A}_2 \end{cases}.$$

Clearly, we have $T(\mathcal{A}_1) \subset \mathcal{A}_2$ and $T(\mathcal{A}_2) \subset \mathcal{A}_1$.

Define $: \mathbb{R}^{+^5} \to \mathbb{R}^+$ by

(18) $$\varphi(t_1, t_2, t_3, t_4, t_5) = \frac{1}{2}\, max\, \left\{t_1, t_2, t_3, \frac{t_4 + t_5}{2}\right\},$$

where $t_1 = d(x,y)$, $t_2 = d(x,Tx)$, $t_3 = d(y,Ty)$, $t_4 = d(x,Ty)$, and $t_5 = d(y,Tx)$, for all $x, y \in X$. It is easy to see that φ satisfies axioms (i) to (iv) of Φ. Then $\varphi \in \Phi$.

Next, we show that T satisfies implicit type cyclic contractive condition. Now, let $(x.y) \in \mathcal{A}_1 \times \mathcal{A}_2$ with $x \ne 0$ and $y \ne 0$. We have

$$\begin{aligned} d(Tx,Ty) = |Tx - Ty| &= \left|\frac{x}{x+4} - \frac{y}{4}\right| \\ &\le \frac{1}{4}(|x| + |y|) \le \frac{1}{4}\left(|x|\left|1 + \frac{1}{x+4}\right| + |y|\left|1 + \frac{1}{4}\right|\right) \\ &= \frac{1}{4}\left(\left|\frac{-x}{x+4} - x\right| + \left|\frac{-y}{4} - y\right|\right) \\ &= \frac{1}{4}(|Tx - x| + |Ty - y|) \end{aligned}$$

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$$\leq \frac{1}{2} \max\{d(x,Tx), d(y,Ty)\}$$
$$\leq \frac{1}{2} \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

Then we have

(19) $\quad d(Tx,Ty) \leq \varphi\big(d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big).$

Moreover, we can show that (19) holds if $x = 0 \; and \; y = 0$.

Now, all conditions of Theorem 2.1 are satisfied (with p = 2), we deduce that T has a unique

$$x_* \in A_1 \cap A_2 = \{0\}.$$

We present another an example showing how our Theorem 2.2 can be used.

Example 2.4. Let $X = \mathbb{R}$ with the usual metric. Suppose $A_1 = [-2,0] = A_3$ and $A_2 = [0,2] = A_4$ and $Y = \cup_{i=1}^{p} A_i$ Define $T: Y \to Y$ such that $Tx = \frac{-x}{4}$ for all $x \in Y$. Clearly, $A_i (i = 1,2,3,4)$ are closed subsets of X. Moreover, mapping T is a implicit type cyclic contractive, where φ : $\mathbb{R}^{+5} \to \mathbb{R}^+$ defined by

(20) $\quad \varphi(t_1, t_2, t_3, t_4, t_5) = At_1 + B[t_2 + t_3] + C[t_4 + t_5],$

Where $t_1 = d(x,y), t_2 = d(x,Tx), t_3 = d(y,Ty), t_4 = d(x,Ty) \; and \; t_5 = d(y,Tx)$, for all $x, y \in X$. Then, $\varphi \in \Phi$.

Indeed, take $A = \frac{1}{2}$ and $B = C = \frac{1}{9}$. Equation (20) is reduces to

$$d(Tx,Ty) = \frac{|x-y|}{4} \leq \frac{1}{2}|x-y| + \frac{1}{9}\left[\frac{5|x|}{4} + \frac{5|y|}{4}\right] + \frac{1}{9}\left[\left|x + \frac{y}{4}\right| + \left|y + \frac{x}{4}\right|\right].$$

To see contractive condition (21) is true, we examine following cases:

Case (I): For $x \in A_1, y \in A_2$.

(i). Suppose $x = -1 \; and \; y = 0$. Then equation (21) holds as it reduces to $\frac{1}{4} < \frac{7}{9}$.

(ii). Suppose $x = 0 \; and \; y = 1$. Then equation (21) holds as it reduces to $\frac{1}{4} < \frac{7}{9}$.

(iii). Suppose $x = -1 \; and \; y = 1$. Then equation (21) holds as it reduces to $\frac{1}{2} < \frac{13}{14}$.

(iv). Suppose $x = -2 \; and \; y = 0$. Then equation (21) holds as it reduces to $\frac{3}{4} < \frac{178}{81}$.

(v). Suppose $x = -2 \; and \; y = 2$. Then equation (21) holds as it reduces to $1 < \frac{13}{3}$.

Case (II): For $x \in A_2, y \in A_1$.

(i). Suppose $x = \frac{1}{2} \; and \; y = \frac{-1}{2}$. Then equation (21) holds as it reduces to $\frac{1}{4} < \frac{13}{18}$.

(ii). Suppose $x = 2 \; and \; y = -1$. Then equation (21) holds as it reduces to $\frac{3}{4} < \frac{178}{81}$.

(iii). Suppose $x = 1 \; and \; y = -1$. Then equation (21) holds as it reduces to $\frac{1}{2} < \frac{13}{14}$.

Case (III): for $x = y = 0, d(Tx, Ty) = 0$. Then equation (21) trivially holds.
Thus in all cases the equation (17) is verified. Similarly other cases can be seen. Hence T is a implicit type cyclic contractive mapping. Therefore, all conditions of Theorem 2.2 are satisfied (with p = 4), and so T has a fixed point (which is $x^* = 0 \in \cap_{i=1}^{4} A_i$.)

In the following, we have some more fixed point results under certain properties.
Theorem 2.3. Let $T: Y \to Y$ be a self-mapping as in Theorem 2.1. Then the fixed point problem for T is well posed, that is, if there exists a sequence $\{y_n\}$ in Y with $d(y_n, Ty_n) \to 0$, as $n \to \infty$, then $y_n \to z$ as $n \to \infty$.
Proof. Owing to Theorem 2.1, we know that for any initial value $y \in Y, z \in \cap_{i=1}^{m} A_i$ is the unique fixed point of T. Thus, $d(y_n, z)$ is well defined. Consider

$$d(y_n, z) \le d(y_n, Ty_n) + d(Ty_n, Tz)$$
$$\le d(y_n, Ty_n) + \varphi\big(d(y_n, z), d(y_n, Ty_n), d(z, Tz), d(y_n, Tz), d(z, Ty_n)\big).$$

Passing to the limit as $n \to \infty$ 1in the above inequality and using property of φ, we have

$$\lim_{n\to\infty} d(y_n, z) \le \varphi\left(\lim_{n\to\infty} d(y_n, z), 0,0, \lim_{n\to\infty} d(y_n, z), \lim_{n\to\infty} d(z, Ty_n)\right).$$

From the property (iv) of φ, we have $d(y_n, z) \to 0$ as $n \to \infty$ which is equivalent to saying that $y_n \to z$ as $n \to \infty$.

Theorem 2.4. Let $T: Y \to Y$ be a self-mapping as in Theorem 2.1. Then T has the limit shadowing property, that is, if there exists a convergent sequence $\{y_n\}$ in Y with $d(y_{n+1}, Ty_n) \to 0$, as $n \to \infty$, then there exists $y \in Y$ such that $d(y_n, T^n x) \to 0$, as $n \to \infty$.
Proof. As in the proof of Theorem 2.3, we observe that for any initial value $y \in Y, z \in \cap_{i=1}^{m} A_i$ is the unique fixed point of T. Thus, $d(y_n, z)$, $d(y_{n+1}, z)$ are well defined. Set y as a limit of a convergent sequence $\{y_n\}$ in Y. Consider

$$d(y_{n+1}, z) \le d(y_{n+1}, Ty_n) + d(Ty_n, Tz)$$
$$\le d(y_{n+1}, Ty_n) + \varphi\big(d(y_n, z), d(y_n, Ty_n), d(z, Tz), d(y_n, Tz), d(z, Ty_n)\big).$$

Passing to the limit as $n \to \infty$ 1in the above inequality and using property of φ, we have

$$d(y, z) \le \varphi\big(d(y, z), 0,0, d(y, z), d(y, z)\big).$$

From the property (iv) of φ, we have $d(y, z) = 0$ as which implies that $y = z$. Thus we have $d(y_n, T^n x) \to d(y, z) = 0$ as $n \to \infty$.

Theorem 2.5. Let X be a non-empty set, (X, d) and (X, ρ) be two metric spaces, $m \in \mathbb{N}$, $A_1, A_2, \dots\dots A_p$ be nonempty closed subsets of X . Then $Y = \cup_{i=1}^{p} A_i$. Suppose that

(1). $Y = \cup_{i=1}^{p} A_i$ is said to be a cyclic representation of Y with respect to T;
(2). $d(x, y) \le \rho(x, y)$ for all $x, y \in Y$;

(3). (Y,d) is a complete metric space;
(4). $T:(Y,d)\to(Y,\rho)$ is continuous
(5). $T:(Y,d)\to(Y,\rho)$ is implicit type cyclic contractive.

Then $\{T^n x_0\}$ converges to z in (Y,d) for any $x_0\in Y$ and z is the unique fixed point of T.

Proof. Let $x_0\in Y$ As in Theorem 2.1, the assumption (5) implies that $\{T^n x_0\}$ is a Cauchy sequence in (Y,ρ) Taking (2) into account, $\{T^n x_0\}$ is a Cauchy sequence in (Y,d) and due to (3) it converges to z in (Y,ρ) for any $x_0\in Y$. Condition (4) implies the uniqueness of z.

3. An application to integral equation

In this section, we apply the result given by Theorem 2.1 to study the existence and uniqueness of solutions to a class of nonlinear integral equations.

We consider the nonlinear integral equation

(22) $u(t)=\int_0^T G(t,s)f\big(s,u(s)\big)ds$ for all $t\in[0,T]$,

Where $T>0, f:[0,T]\times\mathbb{R}\to\mathbb{R}$ and $G:[0,T]\times[0,T]\to[0,\infty)$ are continuous functions.

Let $X=C[0,T]$ be the set of real continuous functions on $[0,T]$. We endow X with the standard metric

$$d_\infty(u,v)=\max_{t\in[0,T]}|u(t)-v(t)|,\ \text{for all } u,v\in X.$$

It is well known that (X,d_∞) is a complete metric space.

Let $(\alpha,\beta)\in X^2, (\alpha_0,\beta_0)\in\mathbb{R}^2$ such that

(23) $$\alpha_0\le\alpha\le\beta\le\beta_0.$$

We suppose that for all $t\in[0,T]$, we have

(24) $$\alpha(t)\le\int_0^T G(t,s)f\big(s,\beta(s)\big)ds$$

and

(25) $$\beta(t)\ge\int_0^T G(t,s)f\big(s,\alpha(s)\big)ds.$$

We suppose that for all $s\in[0,1]$, $f(s,\cdot)$ is a decreasing function, that is,

(26) $$x,y\in\mathbb{R}, x\ge y\Longrightarrow f(s,x)\le f(s,y).$$

We suppose that

(27) $$\sup_{t\in[0,T]}\int_0^T G(t,s)ds\le 1.$$

Finally, we suppose that for all $s\in[0,1]$, for all $x,y\in\mathbb{R}$ with $x\le\beta_0$ and $y\ge\alpha_0$ or $x\ge\alpha_0$ and $y\le\beta_0$,

(28) $|f(s,x)-f(s,y)|\le\varphi\big(d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\big)$,

where $\varphi:[0,\infty)\to[0,\infty)$ is a nondecreasing, continuous function that belongs to Φ.

Now, define the set

$$C=\{u\in C[0,T]:\alpha\le u\le\beta\}.$$

We have the following result.

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Theorem 3.1. Under the assumptions (23)-(28), Problem (22) has one and only one solution $u^* \in C$.

Proof. Define the closed subsets of X, A_1 and A_2 by

$$A_1 = \{\mathrm{u} \in X: u \leq \beta\}$$

and

$$A_2 = \{\mathrm{u} \in X: u \geq \alpha\}.$$

Define the mapping $T: X \to X$ by

$$Tu(t) = \int_0^T G(t,s)f\big(s,u(s)\big)ds \quad \text{for all } t \in [0,T].$$

We shall prove that

(29) $$T(A_1) \subseteq A_2 \text{ and } T(A_2) \subseteq A_1.$$

Let $u \in A_1$, that is, $u(s) \leq \beta(s)$, for all $s \in [0,1]$.

Using condition (26), since $G(t,s) \geq 0$ for all $t,s \in [0,T]$, we obtain that

$$G(t,s)f\big(s,u(s)\big) \geq G(t,s)f\big(s,\beta(s)\big), t,s \in [0,T].$$

The above inequality with condition (24) imply that

$$\int_0^T G(t,s)f\big(s,u(s)\big)ds \geq \int_0^T G(t,s)f\big(s,\beta(s)\big)ds \geq \alpha(t),$$

for all $t \in [0,T]$. Then we have $Tu \in A_2$.

Similarly, let $u \in A_2$, that is, $u(s) \geq \alpha(s)$, for all $s \in [0,T]$.

Using condition (26), since $G(t,s) \geq 0$ for all $t,s \in [0,T]$, we obtain that

$$G(t,s)f\big(s,u(s)\big) \leq G(t,s)f\big(s,\beta(s)\big), t,s \in [0,T].$$

The above inequality with condition (25) imply that

$$\int_0^T G(t,s)f\big(s,u(s)\big)ds \leq \int_0^T G(t,s)f\big(s,\alpha(s)\big)ds \leq \beta(t),$$

for all $t \in [0,T]$. Then we have $Tu \in A_1$. Finally, we deduce that (29) holds.

Now, let $u, v \in A_1 \times A_2$, that is, for all $t \in [0,T]$,

$$u(t) \leq \beta(t), \qquad v(t) \geq \alpha(t).$$

This implies from condition (23) that for all $t \in [0,T]$,

$$u(t) \leq \beta_0, \qquad v(t) \geq \alpha_0.$$

Now, using conditions (27) and (28), we can write that for all $t \in [0,T]$, we have

$$|Tu(t) - Tv(t)| \leq \int_0^T G(t,s)\big|f\big(s,u(s)\big) - f\big(s,v(s)\big)\big|ds$$

$$\leq \int_0^T G(t,s)\varphi(|u(s)-v(s)|, |u(s)-Tu(s)|, |v(s)-Tv(s)|, |u(s)-Tv(s)|, |v(s)-Tu(s)|)ds$$

$$\leq \varphi(d_\infty(u,v), d_\infty(u,Tu), d_\infty(v,Tv), d_\infty(u,Tv), d_\infty(v,Tu)) \int_0^T G(t,s)ds$$

$$\leq \varphi(d_\infty(u,v), d_\infty(u,Tu), d_\infty(v,Tv), d_\infty(u,Tv), d_\infty(v,Tu))$$

This implies that

$$d_\infty(Tv) \le \varphi\big(d_\infty(u,v), d_\infty(u,Tu), d_\infty(v,Tv), d_\infty(u,Tv), d_\infty(v,Tu)\big).$$

Using the same technique, we can show that the above inequality holds also if we take $(u,v) \in A_2 \times A_1$.

Now, all the conditions of Theorem 2.1 are satisfied (with p = 2), we deduce that T has a unique fixed point $u^* \in A_1 \cap A_2 = C$, that is, $u^* \in C$ is the unique solution to (22).

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