**International Journal of Mathematics and Mathematical Sciences**
ISSN: 2454-7271, Copyright@universalprint
Volume: 01 No.: 02, Pages: 123-133, Published: Dec.,2015
Web: www.universalprint.org , Publication: Universal Print
Title Key: Cosmological Models With Linear Equation of State in…

# Cosmological Models with linear equation of state in General Relativity

**A.Y.Shaikh[1] and S.R.Bhoyar[2]**
Department of Mathematics , Indira Gandhi Mahavidyalaya,Ralegaon-445402.India.
*Department of Mathematics , P.N.Mahavidyalaya ,Pusad-445204.India.
e-mail:-shaikh_2324ay@yahoo.com

**Abstract:**
Cosmological models with linear equation of state (EoS) $p=\alpha\rho-\beta$ , where $\alpha$ and $\beta$ are constants, in general relativity have been investigated for Kaluza-Klein universe. By assuming a constant deceleration parameter that leads two different aspects of the volumetric expansion namely a power law and an exponential volumetric expansion, we have obtained the exact solutions of the field equations. Some physical and geometric properties of the models along with physical acceptability of the solutions have also been discussed in detail.

**Keywords:** Kaluza-Klein universe, linear equation of state.

## 1. Introduction

An accelerated expansion of the universe is indicated by recent cosmological observations [1-6]. Our present universe is made up of about $4\%$ ordinary matter, 74% dark energy and 22% dark matter. The accelerated expansion of the universe is due to the negative pressure dubbed as dark energy (DE) with equation of state (EoS) $\omega=\frac{p}{\rho}$.Dark energy is characterized by this EoS parameter which is negative. Equation of state (EoS) parameter which lies close to −1; it could be $\omega=-1$(standard $\Lambda$CDM cosmology),or $\omega>-1$ (the quintessence dark energy) or $\omega<-1$ (phantom dark energy). Babichev, E et. al. [7] analyze the perfect fluid model with a general equation of state $p=\alpha(\rho-\rho_0)$ wherever $\alpha$ and $\rho_0$ are constants. This can be a generalization of a homogenized equation of state ( $\rho_0=0$ ) and is appropriate for the modeling either the linear gas with $p>0$ or the dark energy with $p<0$ . Various relativists [8-12] studied cosmological models with linear equation of state by considering Bianchi space-time in general relativity. Motivating with on high of study work,

**International Journal of Mathematics and Mathematical Sciences**
ISSN: 2454-7271, Copyright@universalprint
Volume: 01 No.: 02, Pages: 123-133, Published: Dec.,2015
Web: www.universalprint.org , Publication: Universal Print
Title Key: Cosmological Models With Linear Equation of State in…

throughout this paper we have a tendency to tend to require under consideration kaluza-klein cosmological model with equation of state (EoS) $p=\alpha\rho-\beta$ , where $\alpha$ and $\beta$ are constants , in General Relativity. Some physical and geometric properties of the models in conjunction with physical satisfactoriness of the solutions have conjointly been mentioned well.

## 2. Metric and Field equations

We consider Kaluza-Klein space-time in the form

$$ds^2=dt^2-A^2\left(dx^2+dy^2+dz^2\right)-B^2d\psi^2, \tag{1}$$

where $A$ and $B$ are functions of $t$ only.

The energy momentum tensor is given by

$$T_\gamma^\mu=diag\left[T_1^1,T_2^2,T_3^3,T_4^4,T_5^5\right]=diag\left[-p,-p,-p,-p,\rho\right], \tag{2}$$

where, $\rho$ is the energy density of the fluid and $p$ is its pressure.

Here we use linear equation of state [13] as

$$p=\alpha\rho-\beta, \tag{3}$$

where, $\alpha$ and $\beta$ are constants.

The Einstein field equations, in natural units ($8\pi G$ = 1 and $c$ = 1), are

$$G_{\mu\gamma}=R_{\mu\gamma}-\frac{1}{2}Rg_{\mu\gamma}=-T_{\mu\gamma}, \tag{4}$$

where, $g_{\mu\lambda}u^\mu u^\gamma=1$ , $u^\mu=(0,0,0,0,1)$ is the five velocity vector, $R_{\mu\gamma}$ is the Ricci tensor, $R$ is the Ricci scalar, and $T_{\mu\gamma}$ is the energy-momentum Tensor.

Using equations (4) , the corresponding field equations for metric (1) with the help of equation (3) and (4) can be written as

$$2\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+2\frac{\dot{A}}{A}\frac{\dot{B}}{B}+\frac{\dot{A}^2}{A^2}=-(\alpha\rho-\beta)\ , \tag{5}$$

$$3\frac{\ddot{A}}{A}+3\left(\frac{\dot{A}}{A}\right)^2=-(\alpha\rho-\beta)\quad , \tag{6}$$

$$3\left(\frac{\dot{A}}{A}\right)^2+3\frac{\dot{A}}{A}\frac{\dot{B}}{B}=\rho\quad , \tag{7}$$

where a dot here in after denotes ordinary differentiation with respect to cosmic time "$t$" only.

**International Journal of Mathematics and Mathematical Sciences**
ISSN: 2454-7271, Copyright@universalprint
Volume: 01 No.: 02, Pages: 123-133, Published: Dec.,2015
Web: www.universalprint.org , Publication: Universal Print
Title Key: Cosmological Models With Linear Equation of State in…

### 3. Solutions of the field equations

Equations (5) and (6) lead to

$$\frac{\ddot{A}}{A}+\frac{2\dot{A}^2}{A^2}-\frac{2\dot{A}\dot{B}}{AB}-\frac{\ddot{B}}{B}=0,$$

$$\frac{d}{dt}\left[\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right]=-\left(\frac{A}{A}-\frac{\dot{B}}{B}\right)\left[\frac{3\dot{A}}{A}+\frac{\dot{B}}{B}\right]. \tag{8}$$

Let $V$ be the function of $t$ defined by

$$V=A^3B. \tag{9}$$

Then from equation (8), we obtain

$$\frac{d}{dt}\left[\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right]=-\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)\frac{\dot{V}}{V}. \tag{10}$$

Integrating the above equation, we get

$$\frac{A}{B}=d\exp\left[x\int\frac{1}{V}dt\right], \tag{11}$$

where $x$ and $d$ are constants of integration.

In view of $V=A^3B$, equation (11) leads to

$$A=d^{\frac{1}{4}}V^{\frac{1}{4}}\exp\left[\frac{x}{4}\int\frac{dt}{V}\right], \tag{12}$$

$$B=d^{\frac{-3}{4}}V^{\frac{1}{4}}\exp\left[\frac{-3}{4}x\int\frac{dt}{V}\right]. \tag{13}$$

Using equations (12) and (13), $A$ and $B$ are explicitly be expressed as

$$A=d_1V^{\frac{1}{4}}\exp\left[X_1\int\frac{dt}{V}\right], \tag{14}$$

$$B=d_2V^{\frac{1}{4}}\exp\left[X_2\int\frac{dt}{V}\right], \tag{15}$$

where

$$d_1^{\,3}d_2=1,\quad 3X_1+X_2=0,\quad d_1=d^{\frac{1}{4}},\quad d_2=d^{-\frac{3}{4}},\quad X_1=\frac{x}{4},X_2=\frac{-3x}{4}.$$

The above equations become

$$A=D_1V^{\frac{1}{4}}\exp\left[X_1\int\frac{dt}{V}\right], \tag{16}$$

$$B = D_2 V^{\frac{1}{4}} \exp\left[ X_2 \int \frac{dt}{V} \right], \tag{17}$$

where $D_1^3 D_2 = 1$ and $3X_1 + X_2 = 0.$

## 4. Isotropization

The isotropy of the expansion can be parametrized after defining the directional Hubble's parameters and the average Hubble's parameter of the expansion. The directional Hubble parameters in the directions $x, y, z, \psi$ for the Kaluza-Klein metric defined in (1) may be defined as follows:

$$H_x = H_y = H_z = \frac{\dot{A}}{A} \text{and } H_\psi = \frac{\dot{B}}{B} \tag{18}$$

The mean Hubble parameter, *H*, is given by

$$H = \frac{\dot{R}}{R} = \frac{1}{4}\frac{\dot{V}}{V} = \frac{1}{4}\left( 3\frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) \tag{19}$$

where *R* is the mean scale factor and $V = R^4 = A^3 B$ is the spatial volume of the universe.
The anisotropy parameter of the expansion $\Delta$ is defined as

$$\Delta = \frac{1}{4}\sum_{i=1}^{4}\left( \frac{H_i - H}{H} \right)^2 \tag{20}$$

in the $x, y, z, \psi$ directions, respectively. The mean anisotropic parameter of the expansion $\Delta$ has a very crucial role in deciding whether the model is isotropic or anisotropic. It is the measure of the deviation from isotropic expansion, the universe expands isotropically when $\Delta = 0$.

Let us introduce the dynamical scalars, such as expansion parameter $(\theta)$ and the shear $(\sigma^2)$ as usual

$$\theta = 4H \tag{21}$$

$$\sigma^2 = 2\Delta H^2 \tag{22}$$

## 5. Analysis and Discussion

Since the field equations (5)–(7) are three equations having four unknowns and are highly nonlinear, an extra condition is needed to solve the system completely. Here we have used two different volumetric expansion laws

$$V = at^b \tag{23}$$

and

$$V = \alpha_1 e^{\beta_1 t}, \tag{24}$$

**International Journal of Mathematics and Mathematical Sciences**
ISSN: 2454-7271, Copyright@universalprint
Volume: 01 No.: 02, Pages: 123-133, Published: Dec.,2015
Web: www.universalprint.org , Publication: Universal Print
Title Key: Cosmological Models With Linear Equation of State in…

where $a$, $b$, $\alpha_1$ ,$\beta_1$ are constants. In this way, all possible expansion histories, the power law expansion, (23), and the exponential expansion, (24), have been covered.

**6. Model for power law**

Using (23) in (16) and (17), we obtain the scale factors as follows:

$$A = D_1\left[a^{\frac{1}{4}}t^{\frac{b}{4}}\right]\exp\left\{\frac{X_1}{a(1-b)}t^{1-b}\right\} \tag{25}$$

and

$$B = D_2\left[a^{\frac{1}{4}}t^{\frac{b}{4}}\right]\exp\left\{\frac{X_2}{a(1-b)}t^{1-b}\right\}. \tag{26}$$

Metric (1) with the help of (25) and (26) can be written as

$$ds^2 = dt^2 - D_1^2 a^{2/4} t^{2b/4}\exp\left\{2\left[\frac{X_1}{a(1-b)}t^{1-b}\right]\right\}\left(dx^2+dy^2+dz^2\right) - D_2^2 a^{2/4}t^{2b/4}\exp\left\{2\left[\frac{X_2}{a(1-b)}t^{1-b}\right]\right\}d\psi^2. \tag{27}$$

At an initial epoch , both the scale factors vanish, start evolving with time and finally as $t \to \infty$ they diverge to infinity. This is consistent with the big bang model. As scale factors diverge to infinity at large time there will be Big rip at least as far in the future which resembles with Katore and Shaikh [14].

Using equations (25) and (26) in equation (7) , we get the energy density as

$$\rho = 3\left\{\frac{b^2}{8t^2} + \frac{b(3X_1+X_2)}{4at^{b+1}} + \frac{X_1^2 + X_1X_2}{a^2t^{2b}}\right\} \tag{28}$$

Using equation (28) and (3) , we obtain the pressure as

$$p = 3\alpha\left\{\frac{b^2}{8t^2} + \frac{b(3X_1+X_2)}{4at^{b+1}} + \frac{X_1^2 + X_1X_2}{a^2t^{2b}}\right\} - \beta \tag{29}$$

The directional Hubble parameters as defined in (18) are found as

$$H_x = H_y = H_z = \frac{b}{4t} + \frac{X_1}{at^b} \tag{30}$$

$$H_\psi = \frac{b}{4t} + \frac{X_2}{at^b} \tag{31}$$

From equation (18), the mean Hubble's parameter, $H$, is given by

$$H = \frac{b}{4t}. \tag{32}$$

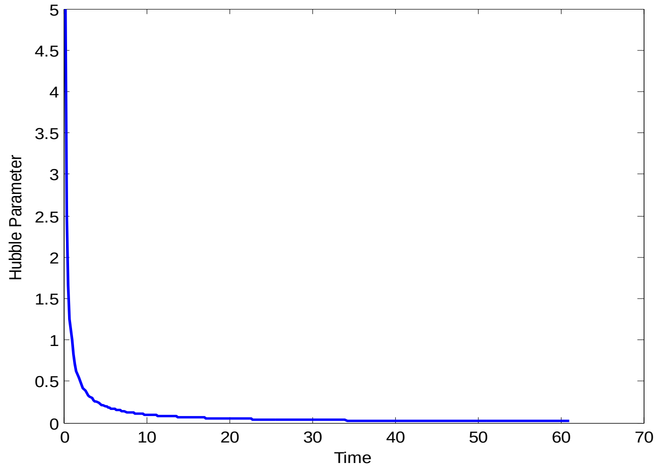


Figure No. 1. Hubble parameter vs Time.

The Hubble parameter decreases with time .

Using the directional and mean Hubble's parameter in (20), we obtain

$$\Delta = \frac{4X^2}{a^2 b^2 t^{2(b-1)}} . \tag{33}$$

From (21) and (22), the dynamical scalars are given by

$$\theta = \frac{b}{t} . \tag{34}$$

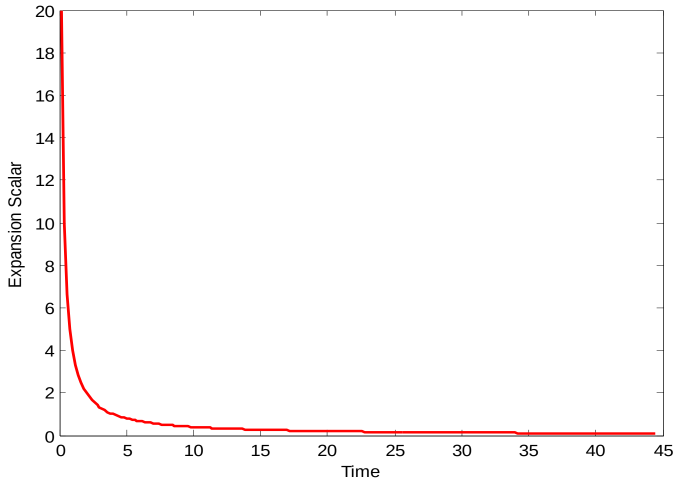


Figure No. 2. Expansion Scalar vs Time.

The shear Scalar

$$\sigma^2 = \frac{X^2}{2a^2 t^{2b}}, \tag{35}$$

where $X^2 = 3X_1^2 + X_2^2 = \text{constant}$.

The deceleration parameter

$$q = \frac{4}{b} - 1. \tag{36}$$

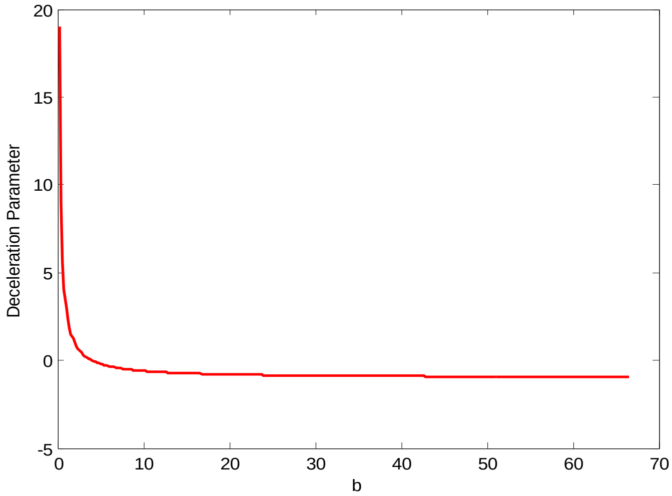


Figure No. 3. Deceleration Parameter vs *b*.

It is mentioned that $q$ was supposed to be positive initially but recent observations from the supernova experiments suggest that it is negative. The positive deceleration parameter corresponds to a decelerating model while the negative value provides inflation. For $b > 4$ the deceleration parameter is negative. The model (27) represents an accelerated universe.

We observe that the Hubble parameter, Expansion Scalar and Shear Scalar are very large at an initial epoch and finally tends to zero as $t \to \infty$. From the value of mean anisotropic parameter in Eq. (33), it is clear that the universe was anisotropic at early stage of evolution and approach to isotropy at large time.

**7. Model for exponential law**

Using (24) in (16) and (17), we obtain the scale factors as follows:

$$A = D_1 \left[ \alpha_1^{\frac{1}{4}} e^{\beta_1 \frac{t}{4}} \right] \exp \left\{ \frac{-X_1}{\alpha_1 \beta_1} e^{-\beta_1 t} \right\} \tag{37}$$

and

**International Journal of Mathematics and Mathematical Sciences**
ISSN: 2454-7271, Copyright@universalprint
Volume: 01 No.: 02, Pages: 123-133, Published: Dec.,2015
Web: www.universalprint.org , Publication: Universal Print
Title Key: Cosmological Models With Linear Equation of State in…

$$B = D_2\left[\alpha_1^{\frac{1}{4}} e^{\beta_1 \frac{t}{4}}\right] \exp\left\{\frac{-X_2}{\alpha_1\beta_1} e^{-\beta_1 t}\right\}. \tag{38}$$

Metric (1) with the help of (37) and (38) can be written as

$$ds^2 = dt^2 - D_1^2\alpha_1^{\frac{1}{2}} e^{\frac{\beta_1 t}{2}} \exp\left\{-2\left[\frac{X_1}{\alpha_1\beta_1} e^{-\beta_1 t}\right]\right\}\left(dx^2 + dy^2 + dz^2\right) - D_2^2\alpha_1^{\frac{1}{2}} e^{\frac{\beta_1 t}{2}} \exp\left\{-2\left[\frac{X_2}{\alpha_1\beta_1} e^{-\beta_1 t}\right]\right\} d\psi^2. \tag{39}$$

The scale factor are constant near $t = 0$, afterwards start increasing with time and as $t \to \infty$, they diverges to infinity. The model is free from singularity. Hence in this case, the volume of the universe is an exponential function which expands with increase in time from constant to infinitely large.

Using equations (37) and (38) in equation (7) , we get the energy density as

$$\rho = 3\left\{\frac{\beta^2}{16} + \frac{\beta(3X_1 + X_2)}{4\alpha_1 e^{\beta_1 t}} + \frac{X_1^2 + X_1X_2}{\alpha_1^2 e^{2\beta_1 t}}\right\} \tag{40}$$

Using equation (40) and (3) , we obtain the pressure as

$$p = 3\alpha\left\{\frac{\beta^2}{16} + \frac{\beta(3X_1 + X_2)}{4\alpha_1 e^{\beta_1 t}} + \frac{X_1^2 + X_1X_2}{\alpha_1^2 e^{2\beta_1 t}}\right\} - \beta \tag{41}$$

The directional Hubble parameters as defined in (18) are found as

$$H_x = H_y = H_z = \frac{\beta}{4} + \frac{X_1}{\alpha_1 e^{\beta_1 t}} \tag{42}$$

$$H_\psi = \frac{\beta}{4} + \frac{X_2}{\alpha_1 e^{\beta_1 t}} \tag{43}$$

From (19), the mean Hubble's parameter, *H*, is given by

$$H = \frac{\beta}{4}. \tag{44}$$

The mean Hubble parameter is constant whereas the directional Hubble parameters are dynamical.

The anisotropy parameter of the expansion, $\Delta$, is

$$\Delta = \frac{4X^2 e^{-2\beta_1 t}}{\alpha_1^2\beta_1^2}. \tag{45}$$

**International Journal of Mathematics and Mathematical Sciences**
ISSN: 2454-7271, Copyright@universalprint
Volume: 01 No.: 02, Pages: 123-133, Published: Dec.,2015
Web: www.universalprint.org , Publication: Universal Print
Title Key: Cosmological Models With Linear Equation of State in…

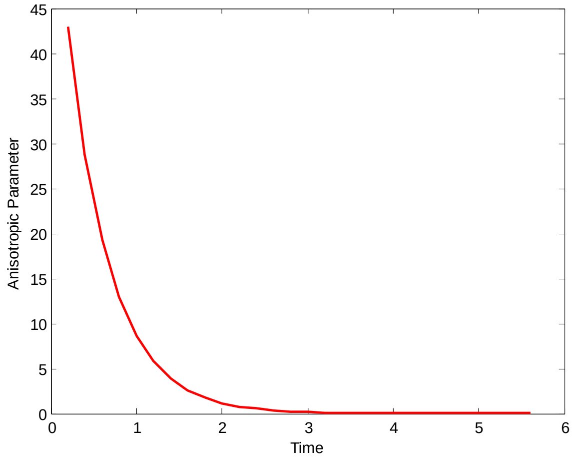


Figure No.4. Anisotropic Parameter vs Time.

At $t = 0$, the anisotropy parameter is constant and decreases with time for $\beta_1 > 0$. It means that the universe was anisotropic at early stage and approaching to isotropy as time increases .

The expansion scalar, $\theta$, is found as

$$\theta = \beta . \tag{46}$$

The rate of expansion of the universe is constant for $\beta > 0$.Thus, the universe evolves with constant rate of expansion.

The shear scalar, $\sigma^2$, is found as

$$\sigma^2 = \frac{X^2 e^{-2\beta_1 t}}{2\alpha_1^2} . \tag{47}$$

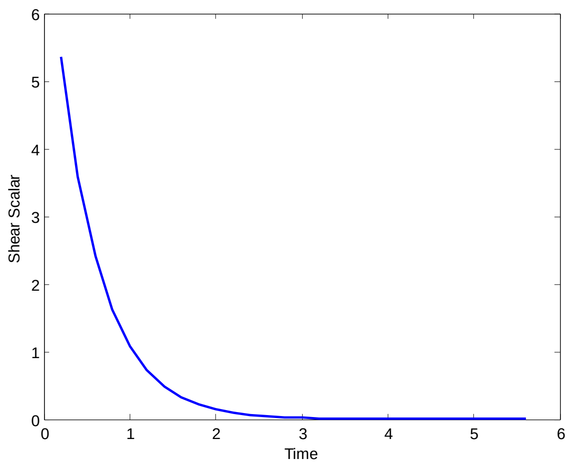


Figure No. 5. Shear Scalar vs Time.

The Shear Scalar $\sigma \to 0$, as $t \to \infty$ .
The deceleration parameter

$$q = -1, \tag{48}$$

where $X^2 = 3X_1^2 + X_2^2 =$ constant .
The sign of $q$ indicate whether the universe accelerates or decelerates. A positive sign of $q$ corresponds to the standard decelerating model and the negative sign of $q$ indicate acceleration. Cosmological observations indicated that the expansion of the universe is accelerating at the present and it was decelerating in the past. Here from Eq. (48), it is observed that the deceleration parameter is negative i.e. the universe is accelerating which is in agreement with current observations of SNe Ia and CMB.

**8. Conclusion:**

We have studied Kaluza-Klein cosmological models with linear equation of state (EoS) $p = \alpha\rho - \beta$ , where $\alpha$ and $\beta$ are constants , in General Relativity. The exact solution of the field equations have been obtained by assuming two different volumetric expansion laws in a way to cover all possible expansion: namely exponential and power law expansion.
In exponential model, the scale factors and the spatial volume increases exponentially as $t$ increases. Since the scalar expansion is a constant, the universe exhibits uniform exponential expansion. We have obtained the deceleration parameter $q = -1$ and $\frac{dH}{dt} = 0$ for this model. Hence, it provides the best values of the Hubble parameter and also the quickest rate of growth of the universe. The ratio of shear scalar to expansion scalar is non zero i.e. the universes is anisotropic and as time increases it tends to zero i.e. at late time the universe tending to isotropy.
In power law model, the scale factors vanish at $t = 0$ and hence the model has the initial singularity. It is observed that the volume of the universe expands indefinitely for all positive values of $b$. The directional Hubble parameters are dynamical. These are diverse at $t = 0$ and approach zero monotonically at $t \to \infty$. The universe starts with an infinite rate of expansion and measure of anisotropy. This is consistent with big bang model.
Also, the models reduce to Strange Quark Matter (SQM) for $\alpha = \frac{1}{3}$ and $\beta = \frac{4}{3}B_c$, where $B_c$ is Bag constant or vacuum energy density of Bag Model of quark matter. It is interesting to note that the results obtained resembles with the investigations of [11,12].

**References:**

[1]. Riess, A.G., et al.: Astron. J. 116, 1009 (1998)
[2]. Perlmutter, S., et al.: Astrophys. J. 517, 565 (1999)
[3]. Tegmark, M., et al.: Phys. Rev. D 69, 103501 (2004)
[4]. Allen, S.W., et al.: Mon. Not. R. Astron. Soc. 353, 457 (2004)
[5]. Spergel, D.N., et al.: Astrophys. J. Suppl. Ser. 148, 175 (2003)
[6]. Komatsu, E., et al.: Astrophys. J. Suppl. Ser. 180, 330 (2009)
[7]. Babichev, E., Dokuchaev, V., Eroshenko, Yu.: arXiv:astro-ph/0407190 (preprint) (2004).
[8]. T. Singh ,R. Chaubey: Int J Theor Phys (2012) 51:90–100.
[9]. K.S. Adhav, I.D. Pawade, A.S. Bansod: International Journal of Mathematical Archive-4(8), 2013, 86-91.
[10]. K.S. Adhav, I.D. Pawade, A.S. Bansod: International Journal of Science and Advanced Technology (ISSN 2221-8386) Volume 3 No 10 October 2013.
[11]. K.S. Adhav, I.D. Pawade, A.S. Bansod: Bulg. J. Phys. 41 (2014) 187–193.
[12]. K. S. Adhav, A. S. Bansod and I. D. Pawade: The African Review of Physics (2014) 9:0003.
[13]. S. Thirukkanesh, S.D. Maharaj :Class. Quant. Grav. 25 235001 , (2008).
[14]. S.D. Katore · A.Y. Shaikh: Astrophys Space Sci (2015) 357:27.