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Solutions of Nonlinear Volterra and Fredholm Integral Equations for Generalized Cyclic Contraction mappings

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Abstract

Consider a class of nonlinear Volterra integral in two variables (0.1) $u(x,y)=f(x,y)+\int_0^x g(x,y,\xi,u(\xi,y))\,d\xi+\int_0^x\int_0^y h\big(x,y,\sigma,\tau,u(\sigma,\tau)\big)d\sigma d\tau$ where f,g,h are given functions and u is the unknown function to be found, and a class of nonlinear Fredholm integro-differential equation of the type (0.2) $x(t)=g(t)+\int_a^b f\big(t,s,x(s),x'(s),\cdots,x^{(n-1)}(s)\big)ds$, for $a\le t\le b$ where x,g,f are real valued functions and $n\ge 2$ is an integer. In this paper we ascertain the existence and uniqueness of solutions for the equations (0.1) and (0.2) for variant of cyclic contractions using fixed point theorems.

Keywords: Fixed point; cyclic contraction; integral equation; integrodifferential equation.

Subject Classification: 47H10, 34B15, 35G30

1. Introduction:

Integral equations arise in many scientific and engineering problems. A large class of initial and boundary value problems can be converted to Volterra or Fredholm integral equations. The potential theory contributed more than any field to give rise to integral equations. Many other applications in science and engineering are described by integral equations or integro-differential equations. The Volterra's population growth model, biological species living together, propagation of stocked fish in a new lake, the heat transfer and the heat radiation are among may areas that are described by integral equations. Many scientific problems give rise to integral equations with logarithmic kernels.

On the other hand, it is well known that discontinuous mappings cannot be (Banach type [1]) contractions. In order to overcome this problem, Kirk et al. [7] introduced cyclic representations and cyclic contractions in the following way. A mapping $\mathcal{T}: \mathcal{A} \cup \mathcal{B} \rightarrow \mathcal{A} \cup \mathcal{B}$ is called cyclic if $\mathcal{T}(\mathcal{A}) \subseteq \mathcal{B}$ and $\mathcal{T}(\mathcal{B}) \subseteq \mathcal{A}$, where $\mathcal{A}, \mathcal{B}$ are nonempty subsets of a metric space (X, d). Moreover, T is called cyclic contraction if there exists $k \in (0,1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x \in A$ and $y \in B$. In [7], some fixed point results were obtained for cyclic contraction mappings. Following this paper, a number of fixed point theorems on (generalized) cyclic contractions have appeared (see, e.g., [2, 3, 14, 12, 5, 6, 8, 9, 13]). It is worth mentioning that a (generalized) cyclic contraction need not be continuous.

The motivation of this paper is originate existence and uniqueness of solutions for a class of nonlinear volterra integral equation in two variables and nonlinear Fredholm integro-differential equation for different variant of cyclic contraction mappings. To accomplish this aspiration the distinguished results given in the papers [8] and [9] are bring into play.

Throughout this paper, we designate the set of all nonnegative real numbers by $\mathbb{R}_+$ and the set of all natural numbers by $\mathbb{N}$.

2. RESULT ON NONLINEAR VOLTERRA INTEGRAL EQUATIONS IN TWO VARIABLES

To complete the theorem of this section, we need following result given by Nashine [8]:

Theorem 2.1 [8, Theorem 2.1] Let (X, d) be a complete metric space, $p \in \mathbb{N}$, $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \cdots, \mathcal{A}_p$ nonempty closed subsets of X and $\mathcal{Y} = \cup_{i=1}^{p} \mathcal{A}_i$. Suppose $\mathcal{T}: \mathcal{Y} \rightarrow \mathcal{Y}$ is a cyclic generalized ψ-weakly contractive mapping, for some $\psi \in \Psi_1$ and $\mathcal{F} \in F_1$. Then $\mathcal{T}$ has a unique fixed point. Moreover, the fixed point of $\mathcal{T}$ belongs to $\cap_{i=1}^{p} \mathcal{A}_i$.

Consider the nonlinear Volterra integral equations in two variables of the forms (Pachpatte [11]):

$$u(x,y) = f(x,y) = \int_0^x g(x,y,\xi,u(\xi,y))\,d\xi + \int_0^x \int_0^y h\big(x,y,\sigma,\tau,u(\sigma,\tau)\big)d\sigma d\tau$$

where f, g, h are given functions and u is the unknown function to be found.

Let $\mathbb{R}$ denote the set of real numbers and $C(S_1, S_2)$ the class of continuous functions from the set S_1 to the set S_2. We denote by $\mathbb{R}_+ = [0,\infty)$, $E = \mathbb{R}_+ \times \mathbb{R}_+$, $E_1 = \{f(x,y,s): 0 \le s \le x < \infty, y \in \mathbb{R}_+\}$; and $E_2 = \{f(x,y,s,t): 0 \le s \le x < \infty,\ 0 \le t \le y < \infty$.

Throughout, we assume that$f \in C(E,\mathbb{R}), g \in C(E_1 \times \mathbb{R},\mathbb{R}), h \in C(E_2 \times \mathbb{R},\mathbb{R})$. Denote by S the space of functions $z \in C(E,\mathbb{R})$ which fulfill the condition

(2.1). $$|z(x,t)| = \mathcal{O}\big(\exp\big(\lambda(x+y)\big)\big),$$
where λ is a positive constant. Define the norm in the space S as

(2.2). $$|z|_S = sup\big[|z(x,t)|\exp\big(-\lambda(x+y)\big)\big].$$

It is easy to see that S with the norm defined in (2.2) is a Banach space. We note that the condition (2.1) implies that there is a constant $M_0 \ge 0$ such that $|z(x,t)| \le M_0 \exp\big(\lambda(x+y)\big)$. Using this fact in (2.2) we observe that

(2.3). $$|z|_S \le M_0.$$
Define a mapping $\mathcal{T}: S \to S$ by

$$(\mathcal{T}u)(x,y) = f(x,y) + \int_0^x g(x,y,\xi,u(\xi,y))\,d\xi + \int_0^x \int_0^y h\big(x,y,\sigma,\tau,u(\sigma,\tau)\big)d\sigma d\tau.$$

for$u \in S$. Note that, if $u^* \in S$ is a fixed point of $\mathcal{T}$, then u^* is a solution of the problem (2.1).

We shall prove the existence of a fixed point of $\mathcal{T}$ under the following conditions.

(I) There exist $\alpha, \beta \in S^2$, $(\alpha_0, \beta_0) \in \mathbb{R}^2$ such that

$$\alpha_0 \le \alpha(x,t) \le \beta(x,t) \le \beta_0(x,t) \in \mathbb{R}_+ \times \mathbb{R}_+$$

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and for all $(x,t) \in \mathbb{R}_+ \times \mathbb{R}_+$, we have

$$\alpha(x,t) = f(x,t) + \int_0^x g(t,s,\xi,\beta(\xi,s))\,d\xi + \int_0^x \int_0^y h\big(t,s,\sigma,\tau,\beta(\sigma,\tau)\big)d\sigma d\tau \text{ and}$$

$$\beta(x,t) = f(x,t) + \int_0^x g(t,s,\xi,\beta(\xi,s))\,d\xi + \int_0^x \int_0^y h\big(t,s,\sigma,\tau,\beta(\sigma,\tau)\big)d\sigma d\tau.$$

(II) The functions g, h in equation (2.1) satisfy the conditions

$$|g(x,y,\xi,u) - g(x,y,\xi,\bar{u})| \le h_1 g(x,y,\xi)|u - \bar{u}|,$$

$$|g(x,y,\sigma,\tau,u) - g(x,y,\sigma,\tau,\bar{u})| \le h_2 g(x,y,\sigma,\tau)|u - \bar{u}|,$$

Where $h_1 \in C(E_1, \mathbb{R}_+)$, $h_2 \in C(E_2, \mathbb{R}_+)$.

(III) There exist nonnegative constants $\delta_1 < 1, \delta_2$ such that

$$\int_0^x h_1(x,y,\xi,\beta(\xi,s)) \exp\big(\lambda(x+y)\big)\,d\xi + \int_0^x \int_0^y h_2(x,y,\sigma,\tau) \exp\big(\lambda(\sigma+\tau)\big)\,d\sigma d\tau < \delta_1 \exp\big(\lambda(x+y)\big),$$

And

$$\left| f(x,y) + \int_0^x g(x,y,\xi,0)d\xi + \int_0^x \int_0^y h_2(x,y,\sigma,\tau,0)d\sigma d\tau \right| < \delta_2 \exp\big(\lambda(x+y)\big),$$

where λ is as given in (2.1).

(IV) The functions g, h in the equation (2.1) satisfy the conditions

$$u, v \in \mathbb{R}, u \le v \Rightarrow g(x,t,\xi,u) \ge g(x,t,\xi,v) \text{ for each } (x,y,\xi) \in E_1, \text{ and}$$

$$h\big(x,t,\sigma,\tau,u(\sigma,\tau)\big) \ge h\big(x,t,\sigma,\tau,\beta(\sigma,\tau)\big) \text{ for each } (x,y,\sigma,\tau) \in E_2$$

(V) There exist $(\alpha, \beta) \in S^2$ such that $\alpha(t) < \beta(t)$ for $t \in \mathbb{R}_+$ and that

$$(\mathcal{T}\alpha)(x,t) \le \beta(x,t) \text{ and } (\mathcal{T}\beta)(x,t) \ge \alpha(x,t) \text{ for } (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+.$$

Theorem 2.2. Under the assumptions (I)-(III), the integral problem (2.1) has a unique solution $u^* \in S$ and it belongs to $\mathcal{P} = \{u \in S: \alpha(x,y) \leq u(x,y) \leq \beta(x,y), (x,y) \in \mathbb{R}_+ \times \mathbb{R}+$.

Proof. The proof of the theorem is divided into three parts.
(A): First we show that $\mathcal{T}$ maps S into itself.

Evidently, $\mathcal{T}u$ is continuous on S and $\mathcal{T}u \in \mathbb{R}$. We verify that (2.1) is fulfilled.

From (2.2), and using conditions (II), (III) and (2.3) we have

(2.4) $$|\mathcal{T}u(x,y)| \leq \left|f(x,y) + \int_0^x g(x,y,\xi,0)d\xi + \int_0^x \int_0^y h_2(x,y,\sigma,\tau,0)d\sigma d\tau\right| + \int_0^x |g(x,y,\xi,u(\xi,y)) - \int_0^x g(x,y,\xi,0)d\xi\, |d\xi + \int_0^x \int_0^y |h(x,y,\sigma,\tau,u(\sigma,\tau)) - h(x,y,\sigma,\tau,0)|\ d\sigma d\tau$$
$$\leq \delta_2 \exp(\lambda(x+y)) + \int_0^x h_1(x,y,\xi)|u(\xi,y)|d\xi + \int_0^x \int_0^y h_2(x,y,\sigma,\tau)|u(\sigma,\tau)|d\sigma d\tau$$
$$\leq \delta_2 \exp(\lambda(x+y)) + |u|_S \left[\int_0^x h_1(x,y,\xi)\exp(\lambda(x+y))\, d\xi + \int_0^x \int_0^y h_2(x,y,\sigma,\tau)\exp(\lambda(\sigma,\tau))\, d\sigma d\tau\right]$$
$$\leq [\delta_2 + M_0\delta_1]\ \exp(\lambda(x+y)).$$

It follows from (2.4) that $\mathcal{T}u \in S$. This proves that $\mathcal{T}$ maps S into itself.
(B): Define closed subsets of S, $\mathcal{A}_1$ and $\mathcal{A}_2$ by
$$\mathcal{A}_1 = \{u \in S: u(x,t) \leq \beta(x,t)\ for\ (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+\}$$
And
$$\mathcal{A}_2 = \{u \in S: u(x,t) \geq \alpha(x,t)\ for\ (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+\}.$$
We shall prove that

(2.5) $$\mathcal{T}(\mathcal{A}_1) \subseteq \mathcal{A}_2\ and\ \mathcal{T}(\mathcal{A}_2) \subseteq \mathcal{A}_1.$$
Let $u \in \mathcal{A}_1$, that is, $u(x,t) \leq \beta(x,t)$ for all $(x,t) \in \mathbb{R}_+ \times \mathbb{R}_+$:
The conditions (I), (IV) and (V) imply that

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$$\mathcal{T}u(x,t) = f(x,t) + \int_0^x g(x,t,\xi,u(\xi,t))d\xi + \int_0^x \int_0^y h\big(x,t,\sigma,\tau,u(\sigma,\tau)\big)d\sigma d\tau$$

$$\geq f(x,t) + \int_0^x g(x,t,\xi,\beta(\xi,t))d\xi + \int_0^x \int_0^y h\big(x,t,\sigma,\tau,\beta(\sigma,\tau)\big)d\sigma d\tau \geq \alpha(x,t)$$

for all $(x,t) \in \mathbb{R}_+ \times \mathbb{R}_+$. Hence, we have$\mathcal{T}u \in \mathcal{A}_2$.
Similarly, if$u \in \mathcal{A}_2$, it can be proved that $\mathcal{T}u \in \mathcal{A}_1$ holds. Thus, (3.11) is fulfilled.

(C): We verify that the operator $\mathcal{T}$ is a cyclic generalized ψ-weakly contractive mapping.
Let $u, v \in \mathcal{A}_1 \times \mathcal{A}_2$, that is, for all $t \in J$,

$$u(x,t) \leq \beta(x,t) \leq \beta_0, \qquad v(x,t) \geq \alpha(x,t) \geq \alpha_0.$$

Using the properties (2.4) of $\mathcal{T}$ and conditions (II) and (III), we conclude that
(2.6) $|\mathcal{T}u(x,y) - \mathcal{T}v(x,y)| \leq \int_0^x |g(x,y,\xi,u(\xi,y)) - g(x,y,\xi,v(\xi,y))|\, d\xi$

$$+ \int_0^x \int_0^y \big|h\big(x,y,\sigma,\tau,u(\sigma,\tau)\big) - h\big(x,y,\sigma,\tau,v(\sigma,\tau)\big)\big| d\sigma d\tau$$

$$\leq \int_0^x h_1\,(x,y,\xi)|u(\xi,y) - v(\xi,y)|d\xi + \int_0^x \int_0^y h_2(x,y,\sigma,\tau)|u(\sigma,\tau) - v(\sigma,\tau)|d\sigma d\tau$$

$$\leq\ |u-v|_S \left[\int_0^x h_1\,(x,y,\xi) \exp\big(\lambda(x+y)\big)\, d\xi + \int_0^x \int_0^y h_2(x,y,\sigma,\tau) \exp\big(\lambda(\sigma,\tau)\big)\, d\sigma d\tau\right]$$

$$< \delta_1 |u-v|_S \exp\big(\lambda(x+y)\big),$$

From (2.6) we obtain (with$k = \delta_1 < 1$)

(2.7) $\quad |\mathcal{T}u - \mathcal{T}v|_S \leq k|u-v|_S \leq k\, max\Big\{|u-v|_S, |u-\mathcal{T}u|_S, |v-\mathcal{T}v|_S, \frac{1}{2}[|u-\mathcal{T}v|_S +$ $v-\mathcal{T}uS]$.

Considering the functions ψ, $\mathcal{F}: [0,+\infty) \to [0,+\infty)$ defined by $\mathcal{F}(t) = t$ and$\psi(t) = (1-k)t$; we get:

$$\mathcal{F}(|\mathcal{T}u - \mathcal{T}v|_S) \leq \mathcal{F}\left(N_\psi(u,v)\right) - \psi\left(\mathcal{F}\left(N_\psi(u,v)\right)\right).$$

Using the same technique, we can show that the above inequality also holds if we take$(u,v) \in \mathcal{A}_2 \times \mathcal{A}_1$,. All other conditions of Theorem 2.1 are fulfilled for the complete metric space $(\mathcal{A}_1 \cup \mathcal{A}_2, |.|_S)$ and $\mathcal{T}$ restricted to $\mathcal{A}_1 \cup \mathcal{A}_2$, (with$p = 2$).

We conclude that the operator $\mathcal{T}$ has a unique fixed point u^* and, hence, the integro-differential equation (2.1) has a unique solution in the set$\mathcal{P}$.

3. RESULT ON NONLINEAR FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

To complete the theorem of this section, we need following result given by Nashine et al. [9]:

Theorem 3.1. [9, Corollary 3.3]. Let (X,d) be a complete metric space,$p \in \mathbb{N}$, $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_p$ nonempty closed subsets of X and $\mathcal{Y} = \cup_{\mathrm{i=1}}^{\mathrm{p}} \mathcal{A}_i$,. Suppose $\mathcal{T}: \mathcal{Y} \to \mathcal{Y}$ is an implicit relation type cyclic contractive mapping, for some$T \in \zeta$. Then $\mathcal{T}$ has a unique fixed point. Moreover, the fixed point of $\mathcal{T}$ belongs to$\mathcal{Y} = \cap_{\mathrm{i=1}}^{\mathrm{p}} \mathcal{A}_i$,.

(3.1) $$x(t) = g(t) + \int_a^b f\left(t, s, x(s), x'(s), \dots, x^{(n-1)}(s)\right) ds,$$

for $a \le t \le b$ where x, g, f are real valued functions and $n \ge 2$ is an integer.

Let $\mathbb{R}$ denote the set of real numbers, $E = \mathbb{R} \times \cdots \times \mathbb{R}$ (n times) be the product space and $I = [a,b]$ $\mathbb{R}_+ = [0,\infty)$ be the given subsets of$\mathbb{R}$. Let $C(S_1, S_2)$ denotes the class of continuous functions from the set S_1 to the setS_2. We assume that$g \in C(I,\mathbb{R})$, $f \in C(I^2 \times E, \mathbb{R})$ and are continuously $(n-1)$-times differentiable with respect to t, on the respective domains of their definitions.

For continuous functions$u^{(j)}(t): I \to \mathbb{R}$ $(j = 0,1,\dots,n-1)$, we denote by $|u(t)|_E = \sum_0^{n-1} u^{(j)}(t)$. Let S be a space of those continuous functions $u(t): I \to \mathbb{R}$ which are $(n-1)$-times continuously differentiable, $u(t), u'(t), \dots, u^{(n-1)}(t) \in E$ and fulfill the condition

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(3.2) $$|u(t)|_E = \mathcal{O}(\exp(\lambda t))$$

where λ is a positive constant. In the space S we define the norm

(3.3) $$|u|_S = \sup_{t\in I}\{\, |u(t)|_E \exp(-\lambda t)\}.$$

It is easy to see that S with the norm defined in (3.3) is a Banach space. We note that the condition (3.2) implies that there exists a nonnegative constant N such that $|u(t)|_E \le N \exp(\lambda t)$. Using this fact in (3.3) we observe that

(3.4) $$|u|_S < N.$$

Define the operator $\mathcal{T}: S \to S$ by

(3.5) $$\mathcal{T}(xt) = g(t) + \int_a^b f(t, s, x(s), x'(s), \dots, x^{(n-1)}(s))\, ds$$

for $x \in S$. Note that, if $x^* \in S$ is a fixed point of $\mathcal{T}$, and then x^* is a solution of the problem (3.1).

We shall prove the existence of a fixed point of $\mathcal{T}$ under the following conditions.

(I) There exist $(\alpha,\beta) \in S^2$, $(\alpha_0,\beta_0) \in \mathbb{R}^2$ such that for $(j = 0,1,\dots,n-1)$

$$\alpha_0 \le \alpha(t) \le \beta(t) \le \beta_0 \text{ and } \alpha_0 \le \alpha^{(j)}(t) \le \beta^{(j)}(t) \le \beta_0, t \in I$$

and for all $t \in I$, we have

$$\alpha(t) \le g(t) + \int_a^b f\left(t, s, \beta(s), \beta'(s), \dots, \beta^{(n-1)}(s)\right) ds$$

$$\alpha^{(j)}(t) \le g^{(j)}(t) + \int_a^t \frac{\partial^j}{\partial t^j} f\left(t, s, \beta(s), \beta'(s), \dots, \beta^{(n-1)}(s)\right) ds, t \in I$$

and

$$\beta(t) \ge g(t) + \int_a^b f\left(t, s, \alpha(s), \alpha'(s), \dots, \alpha^{(n-1)}(s)\right) ds$$

$$\beta^{(j)}(t) \ge g^{(j)}(t) + \int_a^t \frac{\partial^j}{\partial t^j} f\left(t, s, \alpha(s), \alpha'(s), \dots, \alpha^{(n-1)}(s)\right) ds, t \in I.$$

(II) $f: I \times I \times E \to \mathbb{R}$ is continuous and non-increasing with respect to the third and onward up to $(n-1)$ variables, that is, for $u, v \in E$,

$$u(t) \geq v(t) \ and \ u^{(j)}(t) \geq v^{(j)}(t) \ for \ t \in I \implies$$
$$f\left(t, s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) \leq f\left(t, s, v(s), v'(s), \dots, v^{(n-1)}(s)\right), and$$

$$\frac{\partial^j}{\partial t^j} f\left(t, s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) \leq \frac{\partial^j}{\partial t^j} f\left(t, s, v(s), v'(s), \dots, v^{(n-1)}(s)\right),$$
$a \leq s \leq t \leq b.$

(III) The function f and its derivative with respect to t satisfy the conditions

(3.6)
$$\left|\frac{\partial^j}{\partial t^j} f(t, s, u_0, u_1, \dots, u_{n-1}(s)) - \frac{\partial^j}{\partial t^j} f(t, s, v_0, v_1, \dots, v_{n-1}(s))\right| \leq r_j(t,s) \sum_{i=0}^{n-1} |u_i - v_i|,$$

(IV) For $(j = 0,1,\dots,n-1)$ there exist nonnegative constants α_j such that $\sum_{i=0}^{n-1} \alpha_j < 1$ and

(3.7) $$\int_a^b r_j(t,s) \exp(\lambda s)\, ds \leq \alpha_j \exp(\lambda t),$$

fort $\in I$, where λ is as given in (3.2).

(V) for $(j = 0,1,\dots,n-1)$ there exist nonnegative constants P_j such that

$$\left|g^{(j)}(t)\right| + \int_a^t \left|\frac{\partial^j}{\partial t^j} f(t, s, 0,0, \dots, 0)\right| ds \leq P_j \exp(\lambda t)$$

whereg, f are defined in equation (3.1) and λ is as given in (3.2).
We have the following result for the set

$$Q = \{u \in \varepsilon: \alpha(t) \leq u(t) \leq \beta(t), \alpha^{(j)}(t) \leq u^{(j)}(t) \leq \beta^{(j)}(t), t \in I, j = 0,1,\dots,n-1\}.$$

Theorem 3.2. Under the assumptions (I)-(V), the integro-differential problem (3.1) has a unique solution in the set$Q$.

Proof. The proof of the theorem is divided into three steps.
Step-One: First we show that $\mathcal{T}$ maps S into itself.
Differentiating both sides of (3.5) with respect to t we get
(3.9)

$$(\mathcal{T}x)^{(j)}(t) = g^{(j)}(t) + \int_a^t \frac{\partial^j}{\partial t^j} f\left(t, s, x(s), x'(s), \dots, x^{(n-1)}(s)\right) ds$$

for$j = 0,1, \dots, n-1$. Evidently, $(\mathcal{T}x), (\mathcal{T}x)^{(j)}$ are continuous on I and $(\mathcal{T}x)^{(j)} \in \mathbb{R}$. We verify that (3.2) is fulfilled. From (3.5), (3.9) and using the hypotheses and (3.4) we have

$$|(\mathcal{T}x)(t)|_E = \sum_{j=0}^{n-1} \left|(\mathcal{T}x)^{(j)}(t)\right| \leq \sum_{j=0}^{n-1} \left|g^{(j)}(t)\right|$$
$$+ \sum_{j=0}^{n-1} \left| \int_a^b \frac{\partial^j}{\partial t^j} f\left(t, s, x(s), x'(s), \dots, x^{(n-1)}(s)\right) - \frac{\partial^j}{\partial t^j} f(t, s, 0,0, \dots, 0) \right| ds$$
$$+ \sum_{j=0}^{n-1} \int_a^b \left| \frac{\partial^j}{\partial t^j} f(t, s, 0,0, \dots, 0) \right| ds$$
$$\leq \sum_{j=0}^{n-1} P_j \exp(\lambda t)$$
$$+ \sum_{j=0}^{n-1} \int_a^b r_j (t,s) |x(s)|_E ds$$
$$\leq \sum_{j=0}^{n-1} P_j \exp(\lambda t) + |x|_S \sum_{j=0}^{n-1} \int_a^b r_j (t,s) \exp(\lambda s)\, ds$$

$$\leq \left[\sum_{j=0}^{n-1} P_j + N \sum_{j=0}^{n-1} \alpha_j \right] \exp(\lambda t).$$

From (3.10) it follows that$(\mathcal{T}x) \in S$. This proves that $\mathcal{T}$ maps S into itself.

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Step-Two: Define closed subsets ofε, $\mathcal{A}_1$ and $\mathcal{A}_2$ (for$j = 0,1,\dots,n-1$) by

$$\mathcal{A}_1 = \{u \in S: u(t) \leq \beta(t), u^{(j)}(t) \leq \beta^{(j)}(t) \; for \; t \in I\}$$

And

$$\mathcal{A}_2 = \{u \in S: u(t) \geq \alpha(t), u^{(j)}(t) \geq \alpha^{(j)}(t) \; for \; t \in I\}.$$

We shall prove that

(3.11) $\mathcal{T}(\mathcal{A}_1) \subseteq \mathcal{A}_2 \;\; and \;\; \mathcal{T}(\mathcal{A}_2) \subseteq \mathcal{A}_1.$

Let$u \in \mathcal{A}_1$, that is, $u(t) \leq \beta(t), u^{(j)}(t) \leq \beta^{(j)}(t) \; for \; t \in I$. Using condition (II), we obtain that

(3.12) $f\left(t, s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) \geq f\left(t, s, \beta(s), \beta'(s), \dots, \beta^{(n-1)}(s)\right), and$

(3.13) $\frac{\partial^j}{\partial t^j} f\left(t, s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) \geq \frac{\partial^j}{\partial t^j} f\left(t, s, \beta(s), \beta'(s), \dots, \beta^{(n-1)}(s)\right),$

For $a \leq s \leq t \leq b$.

The inequality (3.12) with condition (I) imply that

$$\begin{aligned}(\mathcal{T}u)(t) &= g(t) + \int_a^b f\left(t, s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) ds \\ &\geq g(t) + \int_a^b f\left(t, s, \beta(s), \beta'(s), \dots, \beta^{(n-1)}(s)\right) ds \geq \alpha(t)\end{aligned}$$

for all $t \in I$. The inequality (3.13) with condition (I) imply that

$$\begin{aligned}(\mathcal{T}u)^{(j)}(t) &= g^{(j)}(t) + \int_a^b \frac{\partial^j}{\partial t^j} f\left(t, s, u(s), u'(s), \dots, u^{(n-1)}(s)\right) ds \\ &\geq g^{(j)}(t) + \int_a^b \frac{\partial^j}{\partial t^j} f\left(t, s, \beta(s), \beta'(s), \dots, \beta^{(n-1)}(s)\right) ds \geq \alpha^{(j)}(t)\end{aligned}$$

For all $t \in I$. Hence, we have$\mathcal{T}u \in \mathcal{A}_2$.

Similarly, if$u \in \mathcal{A}_2$, it can be proved that $\mathcal{T}u \in \mathcal{A}_1$ holds. Thus, (3.11) is fulfilled.

Step-Three: We verify that the operator $\mathcal{T}$ is an implicit relation type cyclic contractive map.

Let$(u, v) \in \mathcal{A}_1 \times \mathcal{A}_2$, that is, for all $t \in I$,

$u(t) \le \beta(t) \le \beta_0,\ u^{(j)}(t) \le \beta^{(j)}(t) \le \beta_0,\ \ v(t) \ge \alpha(t) \ge \alpha_0,\ v^{(j)}(t) \ge \alpha^{(j)}(t) \ge \alpha_0.$

Using the properties (3.5) and (3.9) of T and conditions (III), (IV) and (V), we conclude that

$$|(\mathcal{T}x)(t) - (\mathcal{T}y)(t)|_E = \sum_{j=0}^{n-1} \left|(\mathcal{T}x)^{(j)}(t) - (\mathcal{T}y)^{(j)}(t)\right| \le \sum_{j=0}^{n-1} \left|\int_a^b \frac{\partial^j}{\partial t^j} f\left(t, s, x(s), x'(s), \dots, x^{(n-1)}(s)\right) - \frac{\partial^j}{\partial t^j} f\left(t, s, y(s), y'(s), \dots, y^{(n-1)}(s)\right)\right| ds$$

$$\sum_{j=0}^{n-1} \int_a^b r_j\,(t,s)|x(s) - y(s)|_E ds \le |x-y|_S \sum_{j=0}^{n-1} \int_a^b r_j\,(t,s) \exp(\lambda s)\, ds$$

(3.14) $\le |x-y|_S \sum_{j=0}^{n-1} \alpha_j \exp(\lambda t),$

From (3.14) we obtain (with $k = \sum_{j=0}^{n-1} \alpha_j < 1$)

(3.15)

$$|(\mathcal{T}x - \mathcal{T}y)|_S \le k|x-y|_S \le k\, max\left\{|x-y|_S, |x - \mathcal{T}x|_S, |y - \mathcal{T}y|_S, \frac{1}{2}[|x - \mathcal{T}y|_S + |y - \mathcal{T}x|_S]\right\}$$

Consider $T(t_1, \dots, t_6) = t_1 - k \max\left\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\right\}$, where $k \in (0,1)$, then $T \in \mathfrak{T}$ and also

$$T(d(\mathcal{T}x, \mathcal{T}y), d(x,y), d(x, \mathcal{T}x), d(y, \mathcal{T}y), d(x, \mathcal{T}y), d(y, \mathcal{T}x)) \le 0$$

Using the same technique, we can show that the above inequality also holds if we take$(,) \in {}_2 \times {}_1$. All other conditions of Theorem 3.1 are fulfilled for the complete metric space $({}_1 \cup {}_2, |.|)$ and restricted to ${}_1 \cup {}_2$ (with p = 2).

We conclude that the operator has a unique fixed point * and, hence, the nonlinear Fredholm integro-differential equation (3.1) has a unique solution in the set .

7. References

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