International Journal of Universal Sciences and Technology
ISSN: 2464-7263 Volume: 02, Issue: 01, Pages:01-13,
Published:June,2016 Web: www.universalprint.org ,
Title Key: New Coupled Common Fixed Point for Four Mappings…

New Coupled Common Fixed Point for Four Mappings satisfying Rational Contractive Expression

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Abstract

The aim of this paper is to obtain new coupled common fixed point theorems for two pairs of w-compatible self maps satisfying rational contractive condition in metric spaces which is not wholly complete. Our result generalizes and improves related results existing in the literature. Two examples are given in support to show the usability of our results.

Keywords: Coupled fixed point, w-compatible maps, complete metric space.

Subject Classification: 47H10, 54H25.

1. Introduction:

As the Banach contraction principle is a power tool for solving many problems in applied mathematics and sciences, it has been improved and extended in many ways. It is well known that the metric fixed point theory is still very actual, important and useful in all area of Mathematics. It can be applied, for instance in variational inequalities, optimization, dynamic programming, approximation theory, etc.

The well-known Banach contraction theorem plays a major role in solving problems in many branches in pure and applied mathematics. The Banach contraction mapping is one of the pivotal results of analysis. It is a famous tool for solving existence problems in various fields of mathematics. There are a lot of generalizations of the Banach contraction principle in the literature [2, 3, 11]. Ran and Reurings [11] extended the Banach contraction principle in partially ordered sets with some applications to linear and nonlinear matrix equations. Nieto and Rod´riguez-L´opez [10] extended the result of Ran and Reurings and applied their main theorems to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions.

One of the interesting & crucial concepts of fixed point theorem is introduced in 1987 by Guo & Laksmikantham [4] as a notion of coupled fixed point. In 2006 Bhaskar and Lakshmikantham [1] reconsidered the concept of a coupled fixed point of the mapping $F: X \times X \to X$ and investigated some coupled fixed point theorems in partially ordered complete metric spaces. Bhaskar and Lakshmikantham also proved mixed monotone property for the first time and gave their classical coupled fixed point theorem for mapping which satisfy the mixed monotone property. As, an application, they studied the existence and uniqueness of the solution for a periodic boundary value problem associated with first order differential equation. B. S. Choudhury, Meitya and P. Das [2] gave coupled common fixed point theorem for a family of mappings. Many other results on coupled fixed point theory exist in the literature, for more details, we refer the reader to [4, 5, 7, 8, and 10].

In this paper, we obtain a unique common coupled fixed point theorem for two pair of w-compatible self maps satisfying rational contractive condition in metric spaces without considering completeness of whole space. Our result generalizes and improves related results existing in the literature.

Our result generalizes and improves a theorem of Nashine and Zoran et. al. [8] in metric space setting.

2. PRELIMINARIES

First we recall some definitions used throughout the paper.

Definition 2.1. [5]. Let X be a nonempty set and let a mapping$F: X \times X \to X$. An element $(x, y) \in X \times X$ is said to be a coupled fixed point of F if

$$F(x, y) = x \ \ and \ \ F(y, x) = y.$$

It is clear that (x, y) is a coupled fixed point of F if and only if (y, x) is a coupled fixed point of F.

Definition 2.2. [4]. An element $(x, y) \in X \times X$ is called

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a) A coupled coincident point of mappings $F: X \times X \to X$ and $f: X \to X$ if $fx = F(x, y)$ and $fy = F(y, x)$.

b) A common coupled fixed point of mappings $F: X \times X \to X$ and $f: X \to X$ if $x = fx = F(x, y)$ and $y = fy = F(y, x)$.

Definition 2.3. [10] The mappings $F: X \times X \to X$ and $f: X \to X$ are called w-compatible if $f\big(F(x, y)\big) = F(fx, fy)$ *and* $f\big(F(y, x)\big) = F(fy, fx)$ whenever $fx = F(x, y)$ and $fy = F(y, x)$.

3. MAIN RESULTS

The first result of the paper is as follows:

Theorem 3.1. Let (X, d) be a metric space. Let $F, G: X \times X \to X$ and $f, g: X \to X$ be such that

(i) For $x, y, u, v \in X$,

(3.1) $$d\big(F(x, y), G(u, v)\big) \le \frac{\alpha}{2}[d(fx, gu) + d(fy, gv)] + \beta N\big((x, y), (u, v)\big) + \frac{\gamma}{2}\big[d\big(fx, F(x, y)\big) + d(gu, G(u, v)) + d\big(fy, F(y, x)\big) + d(gv, G(v, u))\big]$$

for all $(x, y), (u, v) \in X \times X$. when $D_1 = d\big(fx, G(u, v)\big) + d\big(gu, F(x, y)\big) \neq 0$ *and* $D_2 = d\big(fy, G(v, u)\big) + d\big(gv, F(y, x)\big) \neq 0$, where

(3.2) $$N\big((x, y), (u, v)\big) = \min\left\{\frac{d^2(fx, G(u,v)) + d^2(gu, F(x,y))}{d(fx, G(u,v)) + d(gu, F(x,y))}, \frac{d^2(fy, G(v,u)) + d^2(gv, F(y,x))}{d(fy, G(v,u)) + d(gv, F(y,x))}\right\}$$

and $\alpha, \beta, \gamma \ge 0$ *with* $\alpha + 2\beta + 2\gamma < 1$. Further,

(3.3) $$d\big(F(x_{2n}, y_{2n})G(x_{2n+1}, y_{2n+1})\big) = 0 \text{ if } D_1 = 0 \text{ or } D_2 = 0.$$

Further, suppose

(ii) $F(X \times X) \subseteq g(X)$ and $G(X \times X) \subseteq f(X)$,

(iii) either $f(X)$ or $g(X)$ is a complete subspace of X and

(iv) the pair (F, f) and (G, g) are w- compatible.

Then F, G, f *and* g have a unique common coupled fixed point in $X \times X$. Moreover, the common coupled fixed point of F, G, f *and* g have the form (u, u).

Proof. Let x_0, y_0 be arbitrary points in X. From (ii), there exist sequences $\{x_n\}, \{y_n\}, \{z_n\}$ *and* $\{w_n\}$ in X such that

$$F(x_{2n}, y_{2n}) = gx_{2n+1} = z_{2n},$$
$$F(y_{2n}, x_{2n}) = gy_{2n+1} = w_{2n},$$
$$G(x_{2n+1}, y_{2n+1}) = fx_{2n+2} = z_{2n+1},$$

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and

$$G(y_{2n+1}, x_{2n+1}) = fy_{2n+2} = w_{2n+1}.$$

Now, we claim that, for $n \in \mathbb{N}_0$ for contractive condition,

(3.4) $d(z_{2n+1}, z_{2n}) + d(w_{2n+1}, w_{2n}) \leq \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right)[d(z_{2n}, z_{2n-1}) + d(w_{2n}, w_{2n-1})]$.

Indeed, for n = 1, consider the following possibilities:

Case I: When $z_{2n-1} \neq z_{2n+1}$ *and* $w_{2n-1} \neq w_{2n+1}$. Then $D_1 = d(fx_{2n}, G(x_{2n+1}, y_{2n+1})) + d(gx_{2n+1}, F(x_{2n}, y_{2n})) \neq 0$

and

$D_2 = d(fy_{2n}, G(y_{2n+1}, x_{2n+1})) + d(gy_{2n+1}, F(y_{2n}, x_{2n})) \neq 0$. Hence using **(3.1),** we get:

$$d(F(x_{2n}, y_{2n}), G(x_{2n+1}, y_{2n+1})) \leq \frac{\alpha}{2}[d(fx_{2n}, gx_{2n+1}) + d(fy_{2n}, gy_{2n+1})] + \beta \boldsymbol{N}((x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1}))$$

$$+\frac{\gamma}{2}[d(fx_{2n}, F(x_{2n}, y_{2n})) + d(gx_{2n+1}, G(x_{2n+1}, y_{2n+1})) + d(fy_{2n}, F(y_{2n}, x_{2n})) + dgy2n+1,G(y2n+1, x2n+1),$$

(3.5)

$$d(z_{2n+1}, z_{2n}) = d(F(x_{2n}, y_{2n}), G(x_{2n+1}, y_{2n+1})) \leq \frac{\alpha}{2}[d(z_{2n-1}, z_{2n}) + d(w_{2n-1}, w_{2n})] + \beta \boldsymbol{N}((x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})) + \frac{\gamma}{2}[d(z_{2n-1}, z_{2n}) + d(z_{2n}, z_{2n+1}) + d(w_{2n-1}, w_{2n}) + d(w_{2n}, w_{2n+1})]$$

$$\boldsymbol{N}((x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})) = \min\left\{\frac{d^2(fx_{2n}, G(x_{2n+1}, y_{2n+1})) + d^2(gx_{2n+1}, F(x_{2n}, y_{2n}))}{d(fx, G(x_{2n+1}, y_{2n+1})) + d(gx_{2n+1}, F(x_{2n}, y_{2n}))}, \frac{d^2(fy_{2n}, G(y_{2n+1}, x_{2n+1})) + d^2(gy_{2n+1}, F(y_{2n}, x_{2n}))}{d(fy_{2n}, G(y_{2n+1}, x_{2n+1})) + d(gy_{2n+1}, F(y_{2n}, x_{2n}))}\right\}$$

$$\boldsymbol{N}((x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})) = \frac{d^2(fx_{2n}, G(x_{2n+1}, y_{2n+1})) + d^2(gx_{2n+1}, F(x_{2n}, y_{2n}))}{d(fx, G(x_{2n+1}, y_{2n+1})) + d(gx_{2n+1}, F(x_{2n}, y_{2n}))}$$

(3.6)

$$\boldsymbol{N}((x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})) = d(gx_{2n+1}, fx_{2n}) + d(gx_{2n+1}, fx_{2n+2}) = d(z_{2n}, z_{2n-1}) + d(z_{2n}, z_{2n+1}).$$

On putting the value from eq. **(3.6)** in **(3.5)**, we get

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(3.7)

$$d(z_{2n+1},z_{2n}) \leq \frac{\alpha}{2}[d(z_{2n-1},z_{2n})+d(w_{2n-1},w_{2n})]+\beta[d(z_{2n},z_{2n-1})+d(z_{2n},z_{2n+1})]$$
$$+\tfrac{\gamma}{2}[d(z_{2n-1},z_{2n})+d(z_{2n},z_{2n+1})+d(w_{2n-1},w_{2n})+d(w_{2n},w_{2n+1})].$$

Similarly using that

$$d(w_{2n+1},w_{2n})=d\big(F(y_{2n},x_{2n}),G(y_{2n+1},x_{2n+1})\big)=d\big(G(y_{2n+1},x_{2n+1}),F(y_{2n},x_{2n})\big)$$

and

$$\boldsymbol{N}\big((x_{2n},y_{2n}),(x_{2n+1},y_{2n+1})\big)=\frac{d^2\big(fy_{2n},F(y_{2n+1},x_{2n+1})\big)+d^2\big(gy_{2n+1},F(y_{2n},x_{2n})\big)}{d\big(fy_{2n},G(y_{2n+1},x_{2n+1})\big)+d\big(gy_{2n+1},F(y_{2n},x_{2n})\big)}\Bigg\}$$

$$\boldsymbol{N}\big((x_{2n},y_{2n}),(x_{2n+1},y_{2n+1})\big)=d(w_{2n},w_{2n-1})+d(w_{2n},w_{2n+1})$$

we get

(3.8)

$$d(w_{2n+1},w_{2n}) \leq \frac{\alpha}{2}[d(w_{2n-1},w_{2n})+d(z_{2n-1},z_{2n})]+\beta[d(w_{2n},w_{2n-1})+d(w_{2n},w_{2n+1})]$$
$$+\tfrac{\gamma}{2}[d(w_{2n-1},w_{2n})+d(w_{2n},w_{2n+1})+d(z_{2n-1},z_{2n})+d(z_{2n},z_{2n+1})].$$

Adding **(3.7)** and **(3.8)**, we have

(3.9)

$$d(z_{2n+1},z_{2n})+d(w_{2n+1},w_{2n}) \leq \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right)[d(z_{2n},z_{2n-1})+d(w_{2n},w_{2n-1})].$$

Thus **(3.4)** holds.

Case II: When $z_{2n-1}=z_{2n+1}$ *and* $w_{2n-1}\neq w_{2n+1}$. The first equality implies that $D_1=d\big(fx_{2n},G(x_{2n+1},y_{2n+1})\big)+d\big(gx_{2n+1},F(x_{2n},y_{2n})\big)=0$,and hence $d(z_{2n+1},z_{2n})=d\big(F(x_{2n},y_{2n}),G(x_{2n+1},y_{2n+1})\big)=0$, by **(3.4).** This means that $z_{2n-1}=z_{2n}=z_{2n+1}$. From $w_{2n-1}\neq w_{2n+1}$ as in the first case, we get that **(3.8)** holds true. As a consequence

$$d(w_{2n+1},w_{2n}) \leq \left(\frac{\frac{\alpha}{2}+\beta+\frac{\gamma}{2}}{1-\beta-\frac{\gamma}{2}}\right)d(w_{2n},w_{2n-1}) \leq \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right)d(w_{2n},w_{2n-1}),$$

Since$\left(\frac{\frac{\alpha}{2}+\beta+\frac{\gamma}{2}}{1-\beta-\frac{\gamma}{2}}\right) \leq \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right)$. But then $d(z_{2n},z_{2n-1})=d(z_{2n},z_{2n+1})=0$ implies that **(3.9)** holds.

The case $z_{2n-1} \neq z_{2n+1}$ *and* $w_{2n-1} = w_{2n+1}$ is treated analogously.

Case III: When $z_{2n-1} = z_{2n+1}$ *and* $w_{2n-1} = w_{2n+1}$, then
$D_1 = d(fx_{2n}, G(x_{2n+1}, y_{2n+1})) + d(gx_{2n+1}, F(x_{2n}, y_{2n})) = 0$ and
$D_2 = d(fy_{2n}, G(y_{2n+1}, x_{2n+1})) + d(gy_{2n+1}, F(y_{2n}, x_{2n})) = 0$. Hence **(3.3)** implies that
$z_{2n} = z_{2n+1} = z_{2n+2}$ and $w_{2n} = w_{2n+1} = w_{2n+2}$ and so **(3.9)** holds trivially.

Thus, **(3.4)** holds for n = 1. In a similar way, proceeding by induction, if we assume that **(3.4)** holds, we get that

$$d(z_{2n+2}, z_{2n+1}) + d(w_{2n+2}, w_{2n+1}) \leq \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right)[d(z_{2n+1}, z_{2n}) + d(w_{2n+1}, w_{2n})].$$
$$\leq \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right)[d(z_{2n}, z_{2n-1}) + d(w_{2n}, w_{2n-1})].$$

Hence, by induction, **(3.4)** is proved.

Set $\delta_n := d(z_{2n}, z_{2n-1}) + d(w_{2n}, w_{2n-1}), n \in \mathbb{N}$
and $\Delta := \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right) < 1$. Then, the sequence $\{\delta_n\}$ is decreasing and

$$\delta_n \leq \Delta^n \delta_{n-1} \leq \cdots\cdots\cdots \leq \Delta^n \delta_0.$$

which implies that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty}[d(z_{2n}, z_{2n-1}) + d(w_{2n}, w_{2n-1})] = 0.$$

Thus,

$$\lim_{n\to\infty} d(z_{2n}, z_{2n-1}) = 0 \textit{ and } \lim_{n\to\infty} d(w_{2n}, w_{2n-1}) = 0.$$

It immediately follows; we shall prove that $\{z_{2n}\}$ and $\{w_{2n}\}$ are Cauchy sequences.
For $\delta_n > 0$ and $n \in \mathbb{N}_0$. Then, for each $n \geq m$ we have

$$d(z_{2n}, z_{2m}) \leq d(z_{2n}, z_{2n-1}) + d(z_{2n-1}, z_{2n-2}) + \cdots + d(z_{2m+1}, z_{2m})$$

And

$$d(w_{2n}, w_{2m}) \leq d(w_{2n}, w_{2n-1}) + d(w_{2n-1}, w_{2n-2}) + \cdots + d(w_{2m+1}, w_{2m}).$$

Therefore,

$$d(z_{2n}, z_{2m}) + d(w_{2n}, w_{2m}) \leq \delta_{n-1} + \delta_{n-2} + \cdots + \delta_m$$
$$\leq (\Delta^{n-1} + \Delta^{n-2} + \cdots + \Delta^m)\delta_0$$
$$\leq \frac{\Delta^m}{1-\Delta}\delta_0$$

Letting $n, m \to \infty$, which implies that

$$\lim_{n,m\to\infty}[d(z_{2n}, z_{2m}) + d(w_{2n}, w_{2m})] = 0.$$

Thus $\{z_{2n}\}$ and $\{w_{2n}\}$ are Cauchy sequences in the metric space(X, d). Since $0 \leq \Delta < 1$.

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Hence $\{z_n\}$ and $\{w_n\}$ are Cauchy sequences in the metric space(X,d). Hence we have that $\lim_{n\to\infty} d(z_n,z_m)=0$ *and* $\lim_{n\to\infty} d(w_n,w_m)=0$.

Suppose $f(X)$ is complete. Since $\{z_{2n}\}\subseteq f(X)$ and $\{w_{2n}\}\subseteq f(X)$ are Cauchy sequences in the complete metric space $(f(X),d)$, it follows that the sequence $\{z_{2n}\}$ and $\{w_{2n}\}$ are convergent in $(f(X),d)$. Thus

(3.10) $$\lim_{n\to\infty} d(z_{2n},u)=0$$

And

(3.11) $$\lim_{n\to\infty} d(w_{2n},v)=0$$

for some $u,v \epsilon f(X)$.

Since the pair (F,f) is $w-$compatible, we have $fu=F(u,v)$ and $fv=F(v,u)$.

Suppose that $fu\neq u \text{ or } fv\neq v$,

$$d(fu,u)\le d(fu,z_{2n+1})+\ d(z_{2n+1},u)$$
$$\le d(F(u,v),G(x_{2n+1},y_{2n+1})+\ d(z_{2n+1},u)$$
$$\le \frac{\alpha}{2}[d(fu,z_{2n})+d(fv,w_{2n})]+\beta\boldsymbol{N}\big((u,v),(x_{2n+1},y_{2n+1})\big)$$
$$+\tfrac{\gamma}{2}\big[d\big(fu,F(u,v)\big)+d(z_{2n},G(x_{2n+1},y_{2n+1}))+d\big(fv,F(v,u)\big)+$$

dw2n,G(y2n+1, x2n+1)

where

$$\boldsymbol{N}\big((u,v),(x_{2n+1},y_{2n+1})\big)=\min\left\{\frac{d^2(fu,G(x_{2n+1},y_{2n+1}))+d^2(gx_{2n+1},F(u,v))}{d(fu,G(x_{2n+1},y_{2n+1}))+d(gx_{2n+1},F(u,v))},\right.$$
$$\left.\frac{d^2(fv,G(y_{2n+1},x_{2n+1}))+d^2(gy_{2n+1},F(v,u))}{d(fv,G(y_{2n+1},x_{2n+1}))+d(gy_{2n+1},F(v,u))}\right\}$$
$$=\frac{d^2(fu,G(x_{2n+1},y_{2n+1}))+d^2(gx_{2n+1},F(u,v))}{d(fu,G(x_{2n+1},y_{2n+1}))+d(gx_{2n+1},F(u,v))}$$
$$=d(gx_{2n+1},fu)+d(gx_{2n+1},fu)=2d(gx_{2n+1},fu)=2d(z_{2n},fu)$$

we get

$$d(fu,u)\le\frac{\alpha}{2}[d(fu,z_{2n})+d(fv,w_{2n})]+\beta 2d(z_{2n},fu)$$
$$+\frac{\gamma}{2}\big[d\big(fu,F(u,v)\big)+d(z_{2n},G(x_{2n+1},y_{2n+1}))+d\big(fv,F(v,u)\big)$$
$$+d(w_{2n},G(y_{2n+1},x_{2n+1}))\big].$$

Similarly we have

$$d(fv,v)\le d(fv,w_{2n+1})+\ d(w_{2n+1},v)$$
$$\le d(F(v,u),G(y_{2n+1},x_{2n+1})+\ d(w_{2n+1},v)$$

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$d(fv,v) \leq \frac{\alpha}{2}[d(fu,z_{2n}) + d(fv,w_{2n})] + \beta 2d(w_{2n},fv)$
$+\frac{\gamma}{2}[d(fv,F(v,u)) + d(z_{2n},G(x_{2n+1},y_{2n+1})) + d(fu,F(u,v)) +$
$dw2n,G(y2n+1, x2n+1).$

Hence

$$d(fu,u) + d(fv,v) \leq \alpha[d(fu,z_{2n}) + d(fv,w_{2n})] + 2\beta[d(z_{2n},fu) + d(w_{2n},fv)] + 2\gamma[d(fu,u) + d(fv,v)].$$

Letting $n \to \infty$, and using (3.10) and (3.11), we get

$$d(fu,u) + d(fv,v) \leq (\alpha + 2\beta + 2\gamma)[d(fu,u) + d(fv,v)]$$
$$d(fu,u) + d(fv,v) < d(fu,u) + d(fv,v)$$

which is a contradiction. Hence $fu = u \text{ and } fv = v$. Thus

(3.12)

$$F(u,v) = fu = u \text{ and } F(v,u) = fv = v.$$

Since $F(X \times X) \subseteq g(X)$, there exists $a, b \in X$ such that $u = F(u,v) = ga$ and$v = F(v,u) = gb$.

$$d(u,G(a,b)) = d(F(u,v),G(a,b))$$
$$\leq \frac{\alpha}{2}[d(u,u) + d(v,v)] + \beta \boldsymbol{N}((u,v),(a,b))$$
$$+\frac{\gamma}{2}[d(u,F(u,v)) + d(u,G(a,b)) + d(v,F(v,u)) + d(v,G(b,a))]$$

$d(u,G(a,b)) \leq 0.$

Hence$d(u,G(a,b)) = 0$, which implies that $G(a,b) = u = ga$.

Similarly we have$G(b,a) = v = gb$.

Since the pair (G,g) is $w-$compatible, we have $gu = G(u,v)$ and $gv = G(v,u)$.

Suppose that $gu \neq u \text{ or } gv \neq v$. we have

$d(u,gu) = d(F(u,v),G(u,v))$

$$d(F(u,v),G(u,v)) \leq \frac{\alpha}{2}[d(fu,gu) + d(fv,gv)] + \beta \boldsymbol{N}((u,v),(u,v))$$
$$+\frac{\gamma}{2}[d(fu,F(u,v)) + d(gu,G(u,v)) + d(fv,F(v,u)) +$$
$$dgv,G(v,u)$$
$$\leq \frac{\alpha}{2}[d(u,gu) + d(v,gv)] + 2\beta d(gu,u)$$
$$+\frac{\gamma}{2}[d(u,u) + d(gu,u) + d(v,v) + d(gv,v)]$$

and $d(v,gv) = d(F(v,u),G(v,u))$

$$d(F(v,u),G(v,u)) \leq \frac{\alpha}{2}[d(u,gu) + d(v,gv)] + 2\beta d(gv,v)$$
$$+\frac{\gamma}{2}[d(v,v) + d(gv,v) + d(u,u) + d(gu,u)].$$

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Hence

$$d(u,gu)+d(v,gv)\le(\alpha+2\beta+2\gamma)\,[d(u,gu)+d(v,gv)]$$
$$d(u,gu)+d(v,gv)<\;[d(u,gu)+d(v,gv)]$$

This is a contradiction. Hence $gu=u$ *and* $gv=v$. Thus,

(3.13) $\quad u=gu=G(u,v)$ *and* $v=gv=G(v,u)$.

From **(3.12)** and **(3.13)**, it follows that (u,v) is a common coupled fixed point of F,f,G *and* g.

Let (u^*,v^*) be another common coupled fixed point of F,f,G *and* g. We have

$$d(u,u^*)+d(v,v^*)\le d\big(F(u,v),G(u^*,v^*)\big)+d\big(F(v,u),G(v^*,u^*)\big)$$

$$d\big(F(u,v),G(u^*,v^*)\big)\le\frac{\alpha}{2}[d(u,u^*)+d(v,v^*)]+\beta\boldsymbol{N}\big((u,v),(u^*,v^*)\big)$$
$$+\tfrac{\gamma}{2}\big[d\big(u,F(u,v)\big)+d(u^*,u)+d\big(v,F(v,u)\big)+d(v^*,v)\big]$$

$$d\big(F(u,v),G(u^*,v^*)\big)\le\frac{\alpha}{2}[d(u,u^*)+d(v,v^*)]+2\beta\big(d(u^*,u)\big)+\frac{\gamma}{2}[d(u^*,u)+d(v^*,v)].$$

Similarly we get

$$d\big(F(v,u),G(v^*,u^*)\big)\le\frac{\alpha}{2}[d(u,u^*)+d(v,v^*)]+2\beta\big(d(v^*,v)\big)+\frac{\gamma}{2}[d(u^*,u)+d(v^*,v)].$$

By adding both above inequality, we get

$$d(u,u^*)+d(v,v^*)\le d\big(F(u,v),G(u^*,v^*)\big)+d\big(F(v,u),G(v^*,u^*)\big)$$
$$\le(\alpha+2\beta+2\gamma)[d(u,u^*)+d(v,v^*)]$$
$$<d(u,u^*)+d(v,v^*),$$

which is a contradiction. Hence (u,v) is the unique common coupled fixed point of F,G,f *and* g.

Now we will show that$u=v$. Suppose $u\ne v$.

$$d(u,v)=d\big(F(u,v),G(u,v)\big)$$

$$d\big(F(u,v),G(u,v)\big)\le\frac{\alpha}{2}[d(u,v)+d(v,u)]+\beta\boldsymbol{N}\big((u,v),(v,u)\big)$$
$$+\tfrac{\gamma}{2}[d(u,u)+d(v,u)+d(v,v)+d(u,v)]$$

$d(u,v)<d(u,v)$.

Hence $u=v$.

Thus $u=fu=F(u,u)=G(u,u)=gu=G(u,v)$, that is, the common coupled fixed point of F,G,f *and* g has the form (u,u).

If we take $\gamma=0$ in Theorem 3.1, then we obtain the following corollary.

International Journal of Universal Sciences and Technology
ISSN: 2464-7263 Volume: 02, Issue: 01, Pages:01-13,
Published:June,2016 Web: www.universalprint.org ,
Title Key: New Coupled Common Fixed Point for Four Mappings…

Corollary 3.2. Let (X,d) be a metric space. Let $F,G: X\times X \to X$ and $f,g: X\to X$ be such that

(i) For $x,y,u,v \in X$,

$$d\big(F(x,y),G(u,v)\big) \le \frac{\alpha}{2}[d(fx,gu)+d(fy,gv)] + \beta\boldsymbol{N}\big((x,y),(u,v)\big)$$

for all $(x,y),(u,v)\in X\times X$ and $\alpha,\beta \ge 0$ *with* $\alpha+2\beta<1$.

(ii) $F(X\times X)\subseteq g(X)$ and $G(X\times X)\subseteq f(X)$,

(iii) either $f(X)$ or $g(X)$ is a complete subspace of X and

(iv) the pair (F,f) and (G,g) are w- compatible

Then F,G,f *and* g have a unique common coupled fixed point in $X\times X$. Moreover, the common coupled fixed point of F,G,f *and* g have the form (u,u).

Now a consequence of Theorem 3.1 by Taking $F(x,y)=fx$ *and* $G(u,v)=gu$ where $f: X\to X$ *and* $g: X\to X$, is the following:

Corollary 3.3. Let (X,d) be a metric space. Let $f,g: X\to X$ be mapping such that

(i) For $x,y,u,v\in X$,

$$d(fx,gu) \le \frac{\alpha}{2}[d(x,u)+d(y,v)] + \beta\boldsymbol{N}\big((x,y),(u,v)\big) + \frac{\gamma}{2}[d(x,fx)+d(u,gu)+d(y,fy)+d(v,gv)],$$

for all $(x,y),(u,v)\in X\times X$.

$$\boldsymbol{N}\big((x,y),(u,v)\big) = \min\left\{\frac{d^2(fx,gu))+d^2(gu,fx)}{d(fx,gu))+d(gu,fx))}, \frac{d^2(fy,gv)+d^2(gv,fy))}{d(fy,gv))+d(gv,fy)}\right\}$$

and$\alpha,\beta,\gamma\ge 0$ *with* $\alpha+2\beta+2\gamma<1$.

(ii) $f(X)\subseteq X$ and $g(X)\subseteq X$,

(iii) either $f(X)$ or $g(X)$ is a complete subspace of X

Then f *and* g have a unique common coupled fixed point in$X$.

Remark 3.4 Comparing the conditions in Theorem 3.1 and the conditions in Theorem 2.9 of Nashine and Zoran [8], we see that our result is a generalization of (Theorem 2.9 in [8]) for coupled fixed in metric space for four maps.

4. ILLUSTRATIONS

In this section, we demonstrate our result by the following illustrations:

Example 4.1 Let $X = [0, +\infty)$ then (X, d) is a metric space with the standard metric of real numbers, $d(x, y) = |x - y|$ and mappings $F, G: X \times X \to X$ and $f, g: X \to X$ defined by

$$F(x,y) = \frac{3x^2 - 6y^2}{12},\ G(u,v) = \frac{u^2 - 2v^2}{12}, fx = 3x^2, and\ gu = u^2.$$

It is easy to check that all the condition of Theorem 3.1 are satisfied for $\alpha, \beta, \gamma \geq 0$ with $\alpha + 2\beta + 2\gamma < 1$ and now we will prove that the pair (F, f) and (G, g) are w-compatible.

$$fF(x,y) = 3(\frac{3x^2 - 6y^2}{12})^2 = 3\frac{9x^4 + 36y^4 - 36x^2y^2}{(12)^2} \leq \frac{3x^2 - 6y^2}{12} = F(3x^2, 3y^2)$$
$$= F(fx, fy)$$

and

$$f(F(y,x)) = 3(\frac{3y^2 - 6x^2}{12})^2 = 3\frac{9y^4 + 36x^4 - 36y^2x^2}{(12)^2} \leq \frac{3y^2 - 6x^2}{12} = F(3y^2, 3x^2)$$
$$= F(fy, fx).$$

Then it is clear that F and f are w −compatible. Similarly we can prove that $(G, g)$are w −compatible.

$$g(G(u,v)) = g\left(\frac{u^2 - 2v^2}{12}\right) = \left(\frac{u^2 - 2v^2}{12}\right)^2 \leq \frac{u^4 - 4v^4}{12} = G(u^2, v^2) = G(gu, gv)$$

and

$$g(G(v,u)) = g\left(\frac{v^2 - 2u^2}{12}\right) = \left(\frac{v^2 - 2u^2}{12}\right)^2 \leq \frac{v^4 - 4u^4}{12} = G(v^2, u^2) = G(gv, gu).$$

Then it is clear that G and g are w −compatible.

Now we prove that condition (3.1) is satisfied for $\alpha = \frac{1}{6}, \beta = 0\ and\ \gamma = \frac{1}{6}$ with $\alpha + 2\beta + 2\gamma < 1$.

$$d(F(x,y), G(u,v)) \leq \left|\frac{3x^2 - 6y^2}{12} - \frac{u^2 - 2v^2}{12}\right|$$
$$\leq \frac{1}{12}|(3x^2 - u^2) - 2(3y^2 - v^2)|$$
$$\leq \frac{1}{12}(d(fx, gu) + 2d(fy, gv))$$
$$\leq \frac{2}{12}\frac{(d(fx, gu) + d(fy, gv))}{2}$$

International Journal of Universal Sciences and Technology
ISSN: 2464-7263 Volume: 02, Issue: 01, Pages:01-13,
Published:June,2016 Web: www.universalprint.org ,
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$$\leq \frac{1}{6}\frac{\left(d(fx,gu)+d(fy,gv)\right)}{2}$$
$$\leq \frac{\alpha}{2}[d(fx,gu)+d(fy,gv)]$$
$$\leq \frac{\alpha}{2}[d(fx,gu)+d(fy,gv)]+\beta\boldsymbol{N}\big((x,y),(u,v)\big)$$

$$+\frac{\gamma}{2}\left[d\big(fx,F(x,y)\big)+d(gu,G(u,v))+d\big(fy,F(y,x)\big)+\right.$$
$$dgv,G(v,u).$$

This shows that all the hypothesis of Theorem 3.1 are satisfied. Therefore, we conclude that F,G,f *and* g have a coupled common fixed point in X. This common coupled fixed point is $\big((x,y),(u,v)\big)=(0,0)$.

Example 4.2 Let $X=\mathbb{R}$ be endowed with usual order, $d(x,y)=|x-y|$, and mapping $F,G\colon X\times X\to X$ and $f,g\colon X\to X$ defined by

$$F(x,y)=\frac{3x-4y}{5},\; G(u,v)=\frac{3u-4v}{5}, fx=x, and\ gu=u,$$

Then X satisfies the properties (i) and (ii) and (iii) in Theorem (3.1) except property (iv) for the given mapping.

Now we prove that condition (3.1) is satisfied for $\alpha=\frac{1}{4},\beta=0\ and\ \gamma=\frac{1}{4}$ with $\alpha+2\beta+2\gamma<1$.

$$d\big(F(x,y),G(u,v)\big)\leq\left|\frac{3x-4y}{5}-\frac{3u-4v}{5}\right|$$
$$\leq\frac{3(x-u)}{5}+\frac{4(v-y)}{5}$$
$$\leq\frac{(x-u)}{8}+\frac{(v-y)}{8}+\frac{9(x-u)}{40}+\frac{(x-u)}{40}+\frac{16(v-y)}{40}+\frac{(v-y)}{40}$$
$$\leq\frac{1}{8}\left[(x-u)+(v-y)+\frac{2x+4y}{5}+\frac{2u+4v}{5}+\frac{2y+4x}{5}+\frac{2v+4u}{5}\right]$$
$$\leq\frac{1}{8}[(x-u)+(v-y)]+\frac{1}{8}\left[\frac{2x+4y}{5}+\frac{2u+4v}{5}+\frac{2y+4x}{5}+\frac{2v+4u}{5}\right]$$
$$\leq\frac{\alpha}{2}[d(fx,gu)+d(fy,gv)]$$
$$+\frac{\gamma}{2}\left[d\big(fx,F(x,y)\big)+d(gu,G(u,v))+d\big(fy,F(y,x)\big)+d(gv,G(v,u))\right]$$

It is clear that F,G,f *and* g have a coupled common fixed point in X. This common coupled fixed point is $\big((x,y),(u,v)\big)=(0,0)$.

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