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Existence Results for Caputo Fractional Boundary Value Problems Using Fixed Point Theorems

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Abstract:-
An attempt has been made to establish sufficient conditions for the existence of solutions for a class of Caputo fractional boundary value problems using fixed point theo-rems. Banach fixed point theorem, Schaufer's fixed point theorem and Leray-Schauder type nonlinear alternatve are used to obtain existence results.

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1. Introduction:

Fractional Differential equations have been recently proved to be an effective tool in the modeling of many phenomena in various branches of science and engineering. Numerous applications are found in control systems, visco-elasticity, electrochemistry, pharmacokinetics, food science, etc [1, 2, 3, 19]. Significant contributions in the study of fractional differential equations by researchers has been recorded in the monograph due to Kilbas et al [6]. Some recent results on the theory of fractional differential equations due to Lakshmikantham et. al. can be seen in [7, 8, 9, 10]. Periodic boundary value problem, integral boundary value problem and initial value problem for fractional differential equations of order q, $0 < q < 1$ was studied respectively by Ramirez and Vatsala [20], Wang and Xie [21] and Zhang [22]. Recently, author considered system of fractional differential equations with various type of conditions involving Riemann-Liouville fractional derivative and Caputo fractional derivative of order q, $0 < q < 1$ and obtained existence and uniqueness results via monotone method [4, 11, 12, 13, 14, 15, 16, 17, 18].

An attempt has been made to obtain sufficient conditions for the existence of solution of the following fractional differential equation involving Caputo derivative

$$^{c}D^{q}u(t) = f(t, u(t)) \text{ on } J = [0, T] \quad (1.1)$$

with the boundary conditions

$$u(0) = u_0, \quad u'(0) = u_1 \quad u''(0) = u_2, \quad u'''(0) = u_3 \quad u^{iv}(0) = u_T \quad (1.2)$$

where $^{c}D^{q}$ is the Caputo fractional derivative, $f : J \times R \to R$ is a continuous function and u_0, u_1, u_2, u_3, u_T are real constants. This is called fractional boundary value problem (BVP).

2. Preliminaries

Notation, definitions and preliminary results required in the later section are discussed here. $C(J, R)$ denotes Banach space of all continuous functions from J into R with the norm

$$\|u\|_\infty := \sup\{|u(t)| : t \in J\}.$$

Definition 2.1 [3, 6] The fractional integral of a function u(t) of order q is denoted by $I^q u(t)$. It is defined as

$$I^{q}u(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}u(s)ds$$

Definition 2.2 [3, 6] The Caputo fractional derivative of u(t) of order q is denoted by $^{c}D^{q}u(t)$. It is defined as

$$^{c}D^{q}u(t) = \frac{1}{\Gamma(m-q)}\int_0^t (t-s)^{m-q-1}u(s)ds, m- \leq m, m \in Z+$$

Definition 2.3 A function $u(t) \in C^4(J, R)$ with its q-derivative existing on J is said to be a solution of the problem if u(t) satisfies the equation

$$^{c}D^{q}u(t) = f(t, u(t)) \quad on \quad j = [0, T]$$

and the initial conditions

$$u(0) = u_0, \quad u'(0) = u_1, \quad u''(0) = u_2, \quad u'''(0) = u_3 \quad u^{iv}(0) = u_T.$$

Following Lemmas play important role in the existence of solutions for the BVP (1.1)-(1.2).

Lemma 2.1 [2] Let q > 0, then the fractional differential equation

$$^{c}Dq\, u(t) = 0$$

has solution

$$u(t) = c0 + c1t + c2t2 + c3t3 + ... + c_n t_n = \sum_{i=0}^{n} citi$$

for some ci ∈ R, i = 0, 1, 2, ..., n, n = [q] + 1

Lemma 2.2 [2] Let q > 0, then

I q .cDq h(t) = c0 + c1t + c2t2 + ... + cntn

for some ci ∈ R, i = 0, 1, 2, 3, ..., n, n = [q] + 1

3. Existence Results

Existence result of the BVP (1.1)-(1.2) which is an immediate consequence of Lemma 2.1 and Lemma 2.2.

Lemma 3.1 Let 4 < q ≤ 5 and let u(t) : J → R be continuous. A function u(t) is a solution of the fractional integral equation.

$$u(t) = \frac{1}{\Gamma(q)0}\int_0^t (t-s)^{q-1}u(s)ds - \frac{t^4}{4!\Gamma(q-4)}\int_0^T (T-s)^{q-5}u(s)ds \qquad (3.1)$$

$$+u_0 + u_1 t + \frac{u_2}{2!}t^2 + \frac{u^3}{3!}t^3 + \frac{uT}{4!}t^4$$

if and only if u(t) is a solution of the fractional BVP

cDq u(t) = h(t) t ∈ J (3.2)

$u(0) = u_0$, $u'(0) = u_1$ $u''(0) = u_2$ $u'''(0) = u3$, $u^{iv}(0) = u_T$ (3.3)

Proof: Assume that $u(t)$ satisfies (3.2). Applying Lemma 2.1, we obtain

$$u(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds,$$

$$u'(t) = c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + \frac{1}{\Gamma(q-1)}\int_0^t (t-s)^{q-2}h(s)ds,$$

$$u''(t) = 2c_2 t + 6c_3 t^2 + 4c_4 t^2 \frac{1}{\Gamma(q-2)}\int_0^t (t-s)^{q-3}h(s)ds,$$

$$u'''(t) = 6c_3 + 4.3.2c_4 t \frac{1}{\Gamma(q-3)}\int_0^t (t-s)^{q-4}h(s)ds,$$

$$u^{iv}(t) = 4!c_4 \frac{1}{\Gamma(q-4)}\int_0^t (t-s)^{q-5}h(s)ds,$$

Using initial conditions, we get

$$c_0 = u_0, \quad c_1 = u_1,$$

$$c_2 = \frac{u_2}{2}, \; c_3 = \frac{u_3}{3!},$$

$$c_4 = \frac{u_T}{4!} - \frac{1}{4!\Gamma(q-4)} \int_0^T (T-s)^{q-5} h(s) ds$$

Hence

$$u(t) = u_0 + u_1 t + \frac{u_2}{2!} t^2 + \frac{u_3}{3!} t^3 + \frac{u_4}{4!} t^4 + \frac{1}{4!\Gamma(q-4)} \int_0^T (T-s)^{q-5} h(s) ds$$
$$+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s) ds$$

Conversely, assume that u(t) satisfies fractional integral equation (3.1), then by definition of Caputo derivative, it follows that equation (3.2) and equation (3.3) also holds.

4 Main Results

In this section we obtain results based on Banach fixed point theorem and Schaufer's fixed point theorem.

Following result is obtained by using Banach fixed point theorem.

Theorem 4.1 Assume that there exists a constant k > 0 such that

$$| f(t, y) - f(t, \overline{y}) | \le k \, | y - \overline{y} |$$

for each $t \in J$ *and all* $y, \overline{y} \in \mathbb{R}$. *If*

$$kT^q \left[\frac{1}{\Gamma(q+1)} + \frac{1}{4!\Gamma(q-4)} \right] < 1, \tag{4.1}$$

then BVP (1.1)-(1.2) has a unique solution on J.

Proof: Transform the problem (1.1)-(1.2) into a fixed point problem. Define the operator $F : C(J,R) \to C(J,R)$ *by*

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$$F(u)(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,u)ds - \frac{1}{4!\Gamma(q-4)}\int_0^T (T-s)^{q-5} f(s,u)$$
$$+u_0 + u_1 t + \frac{u_2}{2!}t^2 + \frac{u_3}{3!}t^3 + \frac{u_T}{4!}t^4$$

Clearly, the fixed points of the operator F are solutions of the problem (1.1)-(1.2). We shall use the Banach contraction principle to prove that F has a fixed point. Now, we shall show that F is a contraction mapping. Let $u, v \in C(J,\mathbb{R})$. Then for each $t \in J$, we have

$$|F(u)(t) - F(v)(t)| \le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} | f(s,u(s)) - f(s,v(s))| ds$$
$$+\frac{1}{4!\Gamma(q-4)}\int_0^T (T-s)^{q-5} | f(s,u) - f(s,v)| ds$$
$$\le \frac{k\|u|v\|\infty}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds +$$
$$\frac{k\|u-v\|\infty T^4}{4!\Gamma(q-4)}\int_0^T (t-s)^{q-5}ds$$
$$\le \left[\frac{kT^q}{q\Gamma(q)} + \frac{kT^q}{4!\Gamma(q-4)}\right]\|u-v\|\infty$$
$$= kT^q\left[\frac{1}{\Gamma(q+1)} + \frac{1}{4!\Gamma(q-4)}\right]\|u-v\|\infty$$

Thus

$$\|F(u)(t) - F(v)(t)\|\infty \le kT^q\left[\frac{1}{\Gamma(q+1)} + \frac{1}{4!\Gamma(q-4)}\right]\|u-v\|\infty.$$

Consequently, by equation (4.1), F is a contraction. By Banach fixed point theorem, we claim that F has a fixed point which is a solution of the boundary value problem (1.1)-(1.2).

Following result is based on Schaefer's fixed point theorem:

Theorem 4.2 Assume that

(i) $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous

(ii) There exists a constant M > 0 such that $|f(t, u)| \le M$ for each $t \in J$ *and all* $u \in \mathbb{R}$.

Then the BVP (1.1)-(1.2) has at least one solution on J.

Proof: We shall use Schaufer's fixed point theorem to prove that F has a fixed point. Now we prove:

(a) F **is continuous:**

Let un be a sequence such that $u_n \to u$ in $C(J, \mathbb{R})$. Then for each $t \in J$

$$|F(u_n)(t)-F(u)(t)|\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,u_n(s))-f(s,u(s))|ds$$
$$+\frac{1}{4!\Gamma(q-4)}\int_0^T (T-s)^{q-5}|f(s,u_n)+f(s,u)|ds$$

Since *f* is continuous function, we have

$$\|F(u_n - F(u)\|\infty \to 0 \text{ as } n \to \infty.$$

(b) F **maps the bounded sets into the bounded sets in** C (*J*, R)**:**

It is enough to show that for any $\eta > 0$ there exists positive constant l such that for each $u \in B\eta = \{u \in C(J, \mathbb{R} : \|u\|\infty \le \eta\}$ we have $\|F(u)\|\infty \le l$. By assumption (ii), we have for each $t \in J$

$$|F(u)(t)|\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,u)|ds-\frac{1}{4!\Gamma(q-4)}\int_0^T (T-s)^{q-5}|f(s,u)|ds$$
$$+|u_0|+|u_1|t+|\frac{u_2}{2!}|t^2|+|\frac{u_3}{3!}|t^3+|\frac{u_T}{4!}|t^4$$
$$\le \frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds-\frac{T^4M}{4!\Gamma(q-4)}\int_0^T (T-s)^{q-5}ds+|u_0|+|u_1|T$$
$$+|\frac{u_2}{2!}|T^2+|\frac{u_3}{3!}|T^3+|\frac{u_T}{4!}|T^4$$
$$\le \frac{MT^q}{\Gamma(q+1)}+\frac{MT^q}{\Gamma(q-4)}|u_0|+|u_1|T+|\frac{u_2}{2!}|T^2+|\frac{u_T}{3!}|T^3:=t$$

Thus

$$|F(u)(t)|\le \frac{MT^q}{\Gamma(q+1)}+\frac{MT^q}{\Gamma(q-2)}+|u_0|+|u_1|T+|\frac{u_2}{2!}|T^2+|\frac{u_3}{3!}|T^3|\frac{u_T}{4!}|T^4:=t$$

(c) F **maps bounded sets into the equicontinuous sets** C (*J*, R)**:**

Let $t_1, t_2 \in J$, $t_1 < t_2$, $B\eta$ be bounded set of C (J, R) as in (b) and let $u \in B\eta$. Then

$$|F(u)(t_2)-F(u)(t_1)| = \left|\frac{1}{\Gamma(q)}\int_0^{t_1}\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]|f(s,u(s))|ds\right.$$
$$+\int_{t_1}^{t_2}(t_2-s)^{q-1}f(s,u(s))ds$$
$$+u_1(t_2-t_1)+\frac{u_2}{2!}(t_2^2-t_1^2)+\frac{u_3}{3!}(t_2^3-t_1^3)+\frac{uT}{4!}(t_2^4-t_1^4)$$
$$\le \frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_2-s)^{q-1}]|f(s,u(s))|ds$$
$$+\frac{t_2-t_1}{2\Gamma(q-2)}\int_{t_1}^{t_2}(t_2-s)^{q-1}|f(s,u(s))|ds$$
$$+|u_1|(t_2-t_1)+\frac{|u_2|}{2!}(t_2^2-t_1^2)+\frac{|u_3|}{3!}(t_2^3-t_1^3)+\frac{|uT|}{4!}(t_2^4-t_1^4)$$
$$\le \frac{M}{\Gamma(q+1)}[(t_2-t_1))^q+t_1^q-t_2^q]ds+\frac{M}{2\Gamma(q-1)}(t_2-t_1)^q$$
$$+|u_1|(t_2-t_1)+\frac{|u_2|}{2!}(t_2-t_1)^2+\frac{|u_3|}{3!}(t_2-t_1)^3+\frac{|u_T|}{4!}(t_2^4-t_1^4)$$

As $t1 \to t2$, the right hand side of the above inequality tends to zero. Using Arzela-Ascoli
theroem, we conclude that $F : C(J,\mathrm{R}) \to C(J,\mathrm{R})$ is completely continuous.

(d) A priori bounds:

Now we show that

$$E=\{u\in C(J,\mathbb{R}):u=\lambda F(u)\ for\ some\ 0<\lambda<1\}$$

is bounded.

Let $u \in E$, then $u = \lambda F(u)$ for some $0 < \lambda < 1$. Thus for each $t \in J$ we have

$$u(t)=\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f(s,u(s))ds-\frac{\lambda t^4}{4!\Gamma(q-4)}\int_0^T(T-s)^{q-5}f(s,u(s))ds$$
$$+\lambda u_0+\kappa u_1 t+\lambda\frac{u_2}{2!}t^2+\lambda\frac{u_3}{3!}t^3\lambda\frac{u_T}{4!}t^4$$

This implies by assumption (ii) that for each $t \in J$ we have

$$|u(t)|\le\frac{M}{q\Gamma(q)}T^q+\frac{|M}{(q-4)|\Gamma(q-4)}T^q+|u_0|+|u_0|T+\frac{u_2}{2!}T^2+\frac{u_3}{3!}T^3+\frac{u_T}{4!}T^4$$

Thus for every $t \in J$ we have

$$\|u\|\infty|\le\frac{M}{\Gamma(q+|1)}T^q+\frac{M}{(q-3)}T^q+|u_0|+|u_0|T+\frac{u_2}{2!}T^2+\frac{u_3}{3!}T^3+\frac{u_T}{4!}T^4:=R$$

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This shows that E is bounded. As a consequence of Schaefer's fixed point theorem, we conclude that F has a fixed point which is a solution of the boundary value problem (1.1)-(1.2).

Following existence result for the BVP (1.1)-(1.2) is obtained by using Leray-Schauder type nonlinear alternative.

Theorem 4.3 Assume that

(i) $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous

(ii) There exist $\phi f \in L1(J, \mathbb{R}+)$ and continuous and nondecreasing $\phi : [0, \infty) \to (0, \infty)$ such that $|f(t, u)| \leq \phi f(t)\,\phi(|u|)$ for each $t \in J$ and all $u \in \mathbb{R}$.

(iii) There exists a constant M > 0 such that

$$\frac{M}{\| I^q \phi_f \| L^1 \psi(M) + P + |u_0| + |u_1| T + \frac{|u_2|}{2} T^2 + \frac{|u_T|}{3} T^3 + \frac{|u_T|}{4} T^4} > 1 \qquad (4.2)$$

$$| where P = \frac{T^3}{3} (I^{q-3} \phi f)(T) \psi(M)$$

Then the BVP (1.1)-(1.2) has at least one solution on J.

Proof: Define the operator F as in Theorems 4.1 and 4.2. It can be shown that F is continuous and completely continuous. For $\lambda \in [0, 1]$, let u be such that for each $t \in J$

$$|u(t)| \leq \psi(\|u\| \infty) \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \phi f ds + \frac{\lambda T^4}{4!\Gamma(q-4)} \int_0^T (T-s)^{q-5} \phi_f ds$$
$$+ \lambda |u_0| + |u_1| T + \frac{|u_2|}{2!} T^2 + \frac{|u_3|}{3!} T^3 + \frac{|u_T|}{4!} T^4$$

$$\frac{\|u\| \infty}{\psi(\|u\| \infty) \| I^q \phi_f \| L^1 + \Delta + |u_0| + |u_1| T + \frac{|u_2|}{2} T^2 + \frac{|u_3|}{3} T^3 + \frac{|u_T|}{4} T^4} \leq 1$$

Where $\Delta = \frac{T^3}{3} (I^{q-3} \phi_f)(T) \psi(\|u\| \infty)$ Then by inequality (4.2), there exists M such that $\|u\| \infty \neq M$

Let

$$Y = \{u \in C(J, \mathbb{R}) : \|u\| \infty \leq M\}$$

The operator $F : \overline{Y} \to C(J,\mathbb{R})$ is continuous and completely continuous. By the choice of Y, there exists no $u \in \partial Y$ such that $u = \lambda F(u)$ for some $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [5], we deduce that F has a fixed point u in $\overline{Y}$, which is the solution of the BVP (1.1)-(1.2).

References

[1] R. P. Agarwal, M. Benchora, S. Hamani, *Boundary Value Problems for Fractional Differential Equations*, Georgian Mathematical Journal, 16(3) (2009), 401-411.

[2] M. Benchora, S. Hamani, S. K. Ntouyas, *Boundary Value Problems for Differential Equations with Fractional Order and Nonlocal Conditions*, Nonl. Anal. 71 (2009), 2391-2396.

[3] M. Caputo, *Linear Models of Dissipition whose Q is Almost Independent, II*, Geophy. J. Roy. Astronom. 13 (1967), 529-539.

[4] D. B. Dhaigude, J. A. Nanware and V. R. Nikam, *Monotone Technique for System of Caputo Fractional Differential Equations with Periodic Boundary Conditions*, Dyn. Conti. Dis. Impul. Sys. A, 19 (5a)(2012), 575-584.

[5] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.

[6] A. A. Kilbas, H.M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North Holland Mathematical Studies Vol.204. Elsevier (North-Holland) Sciences Publishers, Amsterdam, 2006.

[7] V. Lakshmikantham, A. S. Vatsala, *Theory of Fractional Differential Equations and Applications,* Commun.Appl.Anal., 11 (2007) 395-402.

[8] V. Lakshmikantham, A. S. Vatsala, *Basic Theory of Fractional Differential Equations and Applications,* Nonl. Anal., 69(8)(2010), 2677-2682.

[9] V. Lakshmikantham, A. S. Vatsala, *General Uniqueness and Monotone Iterative Technique for Fractional Differential Equations,* Appl. Math. Lett., 21(8), (2008) 828-834.

[10] V. Lakshmikantham, S. Leela and J. V. Devi, *Theory of Fractional Dynamic Systems,* Cambridge Scientific Publishers, Cambridge, U.K , 2009.

[11] J. A. Nanware, *Monotone Method in Fractional Differential Equations and Applications,* Ph.D Thesis, Dr.Babasaheb Ambedkar Marathwada University, 2013.

[12] J. A. Nanware, *Existence and Uniqueness Results for Fractional Differential Equations Via Monotone Method*, Bull. Marathwada Math. Soc., 14(1), (2013) 39-56.

[13] J. A. Nanware, D. B. Dhaigude, *Existence and Uniqueness of solution of Riemann-Liouville Fractional Differential Equations with Integral Boundary Conditions*, Int. Jour. Nonlinear Science. 14 (4)(2012), 410-415.

[14] J. A. Nanware, D. B. Dhaigude, *Monotone Iterative Scheme for System of Riemann- Liouville Fractional Differential Equations with Integral Boundary Conditions*, Math. Modelling Scien. Comput., Springer-Verlag 283 (2012), 395-402.

[15] J. A. Nanware, D. B. Dhaigude, *Monotone Technique for Finite System of Caputo Fractional Differential Equations with Periodic Boundary Conditions*, Dyn. Conti. Dis. Impul. Sys. A, 22(1a), 13-23 (2015).

[16] J.A.Nanware, D.B.Dhaigude, *Boundary Value Problems for Differential Equations of Non-integer Order Involving Caputo Fractional Derivative,* Adv. Stu. Contem. Math., 24(3), 369-376 (2014).

[17] J.A.Nanware, D.B.Dhaigude, *Existence and Uniqueness of Solution of Differential Equations of Fractional Order with Integral Boundary Conditions*, J. Nonlinear. Sci. Appl., 7, 246-254 (2014).

[18] J.A.Nanware, N.B.Jadhav, D.B.Dhaigude, *Monotone Iterative Technique for Finite System of Riemann-Liouville Fractional Differential Equations with Integral Boundary Conditions*, International Conference of Mathematical Sciences 2014, Elsevier, 235-238 (2014).

[19] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[20] J. D. Ramirez, A. S. Vatsala, *Monotone Iterative Technique for Fractional Differential Equations with Periodic Boundary Conditions*, Opus. Mathe., 29 (3)(2009), 289-304.

[21] T. Wang, F. Xie, *Existence and Uniqueness of Fractional Differential Equations with Integral Boundary Conditions*, The J. Nonl. Sci. Appl., 1 (4)(2009), 206-212.

[22] S. Zhang, *Monotone Iterative Method for Initial Value Problems Involving Riemann- Liouville Fractional Derivatives*, Nonl. Anal., 71 (2009) 2087-2093.