**International Journal of Universal Mathematics and Mathematical Sciences**
ISSN: 2454-7271 Volume: 02, Issue: 01, Pages:01-13,
Published:June,2016 Web: www.universalprint.org ,
Title Key: Monotone Method For Finite Difference Equations…

# Monotone Method for Finite Difference Equations Of Reaction Diffusion and Applications

**R. M. Dhaigude**
Department of Mathematics,
Government Vidarbha Institute of Science and Humanities,
Amaravati- 444 604 (M. S.) INDIA.
Email : *rmdhaigude@gmail.com*

**Abstract**

The purpose of this paper is to develop monotone iteration scheme using the notion of upper and lower solutions of nonlinear finite difference equations,which corresponds to the nonlinear reaction diffusion equations with linear boundary conditions. Two monotone sequences are constructed for the finite difference equations when two sequences converge monotonically from above and below to maximal and minimal solutions, which leads to the Existence-Comparison and Uniqueness results for the solution of the nonlinear finite difference system. Positivity Lemma is the main ingredient used in the proof of these results.

**Keywords :** Monotone Scheme, Monotone Property, Existence-Comparison Theorem, Uniqueness Theorem.

**Mathematics Subject Classification :** Ams: 65 No 5; CR; G 1.8

## 1. Introduction

Various real problems in different fields from science and technology are governed by nonlinear reaction diffusion equations. The method of upper and lower solutions is one of the well known method employed successfully in the study of existence-comparison and uniqueness of solutions of IBVP of a nonlinear partial differential equations. In 1992,Sattinger [8] first developed this method for nonlinear parabolic as well as elliptic boundary value problems. An excellent account of these results are given in the elegant books by Ladde, Lakshmikantham and Vatsala [5] and Pao [7]. In the year 1985, Pao [6] developed this method for finite difference equations of nonlinear parabolic and elliptic boundary value problems. Here, we develop the monotone scheme for finite difference system of nonlinear time degenerate parabolic problems.

**We plan the paper as follows:**

In section 2, finite difference system of nonlinear time degenerate parabolic initial boundary value problem is formulated from the corresponding continuous problem under consideration. Section 3 is devoted for the monotone scheme for the discrete problem. Using upper and lower solutions as distinct initial iterations, two monotone sequences are constructed, which converge monotonically from above and below to maximal and minimal solutions respectively. Also the existence-comparison **and uniqueness results for discrete problems are discussed in the last section.**

## 2. Finite Difference Equations:

Consider the time degenerate Dirichlet initial boundary value problem

$d(x, t)u_t - L[u] = f(x, t, u)$; in $D_T$

Boundary condition $u(x, t) = h(x, t)$; on $S_T$ (2.1)

Initial condition $u(x, 0) = \psi(x)$; in $\Omega$

This equation can be written as

$d(x, t)u_t - L[u] + \underline{c}(x, t)u = \underline{c}(x, t)u + f(x, t, u)$; in $D_T$

$u(x, t) + \underline{b}(x,t)u = \underline{b}(x,t)u + h(x, t)$; on $S_T$ (2.2)

$u(x, 0) = \psi(x)$; in $\Omega$

where $\Omega$ is a bounded domain in $IR_P$ ($P = 1, 2, \ldots\ldots$)with boundary $\partial\Omega$; $D_T$ , $S_T$ ,

$$T > 0,\ L[u] = \sum_{ij=1}^{n} a_{i.j}(x,t)\frac{\partial^2 u}{\partial k_i \partial k_j} + \sum_{j=1}^{n} b_j(x,t)\frac{\partial u}{\partial x_j}$$

i.e. $L[u] = D(x, t)\nabla^2 u + b(x, t) . \nabla u$ and

$$b(x, t) . \nabla u = b^{(1)}(x,t)\ \frac{\partial u}{\partial x_1} + \ldots\ldots + b^{(p)}(x,t)\frac{\partial u}{\partial x_p}.$$

Note that $D(x, t) > 0$ and $b(x, t)$ are diffusion and convection coefficients respectively on $D_T$. Now, we write the discrete version of the above continuous time degenerate Dirichlet IBVP (2.1) by converting it into finite difference equations.

Suppose that $i = (i_1, i_2, i_3, \ldots i_P)$ is a multiple index with $i_v = 0, 1, 2, ., M_{v+1}$ and $x_i = (x_{i1}, x_{i2}, ., x_{ip})$ is a arbitrary mesh point in $\Omega_p$ where $M_v$ is the total number of interior mesh points in the $x_{iv}$ coordinate direction. Denote by $\Omega_p$, $\bar{\Omega}_p$, $\partial\,\Omega_p$, $\Lambda p$ and $S_p$ the sets of mesh points in $\Omega, \bar{\Omega}, \partial\Omega,$ and ; $D_T$ , $S_T$ respectively and $\bar{\Lambda}_p$, denote the set of all mesh points. Suppose $(i, n)$ is used to represent the mesh point $(x_i, t_n)$.

Set $u_{i,n} \equiv u(x_i, t_n)$; $f_{i,n}(u_{i,n}) \equiv f(x_i, t_n, u(x_i, t_n))$; $g_{i,n}(u_{i,n}) \equiv g(x_i, t_n, u(x_i, t_n))$;
$D_{i,n} \equiv D(x_i, t_n)$ $b_{i,n} \equiv b(x_i, t_n)$; $\psi_i \equiv \psi(x_i)$; $u_{i,0} \equiv u(x_{i,0})$; $d_{i,n} \equiv d(x_i, t_n)$.

Suppose $k_n = t_n - t_{n-1}$ is the $n^{th}$ time increment for $n = 1, 2, \ldots, N$ and $h_v$ is the special increment in the $xi_v$ coordinate direction. Suppose $C_v = (0, \ldots\ldots 1, \ldots\ldots 0)$ is the unit vector in $R^p$ where the constant 1 appears in the $v^{th}$ component and zero elsewhere. Using the standard second order difference approximations. We have

$$\Delta^v u_{i,n} = h_v^{-2}\left[u(x_i + h_v e_v, t_n) - 2u(x_i, t_n) + u(x_i - h_v e_v, t_n)\right]$$

Also we have the standard forward and backward first order difference approximations

$$\delta_+^{(v)} u_{i,n} = h_v^{-1}\left[u(x_i + h_v e_v, t_n) - u(x_i, t_n)\right] \text{ and}$$
$$\delta_-^{(v)} u_{i,n} = h_v^{-1}\left[u(x_i, t_n) - u(x_i - h_v e_v, t_n)\right] \text{ respectively.}$$

In order to avoid the technical difficulties construction of monotone sequences, suppose that each component $b^l(x_i, t_n)$, $l = 1, 2, \ldots, p$ has the same sign in $\Omega_p$ but boundary difference possess different signs for different $\ell$ .Then we can define the first order approximations in the Xiv coordinate direction as

$$\delta^{(v)} u_{i,n} = \begin{cases} \delta_+^{(v)} u_{i,n} & \text{when } b_{i,n}^{(v)} \geq 0 \\ \delta_-^{(v)} u_{i,n} & \text{when } b_{i,n}^{(v)} \leq 0 \end{cases}$$

Now the discrete version of the continuous problem
(2.1) is given by

$$£\,[u_{i,n}] \equiv d_{i,n}k_n^{-1}(u_{i,n} - u_{i,n-1}) - L[u_{i,n}] = f_{i,n}(u_{i,n}); \quad (i, n) \in \Lambda_p. \tag{2.3}$$
$$u_{i,n} = g_{i,n}(u_{i,n}); \qquad (i, n) \in S_p$$
$$u_{i,0} = \psi_i; \quad i \in \Omega_p$$

This problem (2.3) can be written as

$$d_{i,n}k_n^{-1}(u_{i,n} - u_{i,n-1}) - L[u_{i,n}] + \underline{c}_{i,n}u_{i,n} = \underline{c}_{i,n}u_{i,n} + f_{i,n}(u_{i,n}); \text{ in } \Lambda_p$$
$$u_{i,n} + b_{i,n}u_{i,n} = b_{i,n}u_{i,n} + g_{i,n}(u_{i,n}); \ (i,n)\in S_p \tag{2.4}$$
$$u_{i,0} = \psi_i; \qquad i\in\Omega_{pc} \text{ where } L[u_{i,n}] = \sum_{v=1}^{p}\left(D_{i,n}\Delta^{(v)}u_{i,n} + b_{i,n}^{(v)}\delta^{(v)}u_{i,n}\right).$$

Assume that

(i) The coefficient $d_{i,n}$ is a non-negative in $\Lambda_p$. However we will not assume that $d_{i,n}$ is a bounded away from zero. Since $d_{i,n} = 0$ for some $(i, n) \in \Lambda_p$ and hence the equation is time degenerate,

(ii) The functions $f_{i,n}(u_{i,n})$; $g_{i,n}(u_{i,n})$, $\psi_i$ and $u_{i,n}$ are Holder continuous functions in their respective domains,

**International Journal of Universal Mathematics and Mathematical Sciences**
ISSN: 2454-7271 Volume: 02, Issue: 01, Pages:01-13,
Published:June,2016 Web: www.universalprint.org ,
Title Key: Monotone Method For Finite Difference Equations…

(iii) $f_{i,n}(u_{i,n})$ and $g_{i,n}(u_{i,n})$ satisfies the Lipschitz condition $u_{i,n}$ in $\Lambda$p

$$\begin{aligned} -\underline{c}_{i,n}\left(u_{i,n}^{(1)}-u_{i,n}^{(2)}\right) &\le f_{i,n}\left(u_{i,n}^{(1)}\right)-f_{i,n}\left(u_{i,n}^{(2)}\right)\le \bar{c}_{i,n}\left(u_{i,n}^{(1)}-u_{i,n}^{(2)}\right) \\ -\underline{b}_{i,n}\left(u_{i,n}^{(1)}-u_{i,n}^{(2)}\right) &\le g_{i,n}\left(u_{i,n}^{(1)}\right)-g_{i,n}\left(u_{i,n}^{(2)}\right)\le \bar{b}_{i,n}\left(u_{i,n}^{(1)}-u_{i,n}^{(2)}\right)\end{aligned} \quad \text{for} \quad \hat{u}_{i,n}\le u_{i,n}^{(2)}\le u_{i,n}^{(1)}\le \tilde{u}_{i,n},$$

(iv) $F_{i,n}(u_{i,n}) \equiv c_{i,n}u_{i,n} + f_{i,n}(u_{i,n})$ is a monotone nondecreasing in $u_{i,n}$ for $u_{i,n}\in<\hat{u}_{i,n}\tilde{u}_{i,n}>$ and satisfies the Lipschitz condition $|F_{i,n}\left(u_{i,n}^{(1)}\right)-F_{i,n}\left(u_{i,n}^{(2)}\right)|\le k_{i,n}^{(1)}\left(u_{i,n}^{(1)}-u_{i,n}^{(2)}\right)$ for $u_{i,n}^{(1)},u_{i,n}^{(2)}\in<\hat{u}_{i,n},\tilde{u}_{i,n}>$. where $k_{i,n}^{(1)}$ is a constant and independent of $(i, n)$,

(v) The function $G_{i,n}(u_{i,n}) = b_{i,n}u_{i,n} + g_{i,n}(u_{i,n})$ is a monotone nondecreasing in $u_{i,n}$, for $u_{i,n}\in<\hat{u}_{i,n},\tilde{u}_{i,n}>$ and satisfies the Lipschitz condition $|G_{i,n}\left(u_{i,n}^{(1)}\right)-G_{i,n}u_{i,n}^{(2)}\le K_{i,n}^{(2)}|\left(u_{i,n}^{(1)}-u_{i,n}^{(2)}\right)$ For $u_{i,n}^{(1)},u_{i,n}^{(2)}\in<\hat{u}_{i,n},\tilde{u}_{i,n}>$. In terms of $F$ and $G$ the problem (2.4) can be written as

$$\begin{aligned} &\pounds[u_{i,n}] = F_{i,n}(u_{i,n});\ (i, n)\in\Lambda_p \\ &u_{i,n} + \underline{b}_{i,n}u_{i,n} = h_{i,n}(u_{i,n}); \qquad (i, n)\in S_p \\ &u_{i,0} = \psi_i; \qquad i\in\Omega_p \end{aligned} \tag{2.5}$$

where $\pounds[u_{i,n}] \equiv d_{i,n}k_n^{-1}[u_{i,n} - u_{i,n-1}] - L[u_{i,n}] + c_{i,n}u_{i,n}$.

The system (2.3) gives a suitable finite difference approximation for the construction of monotone sequences.

To prove our main results we develop monotone scheme for the finite difference equation (2.2). The following positivity lemma is a discrete version of the positivity lemma for the continuous problem.

**Lemma 2.1 (Positivity Lemma)** Suppose that $u_{i,n}$ satisfies the following inequalities

$$\begin{aligned} &d_{i,n}k_n^{-1}(u_{i,n} - u_{i,n-1}) - L[u_{i,n}] + c_{i,n}u_{i,n}\ge 0 \quad \text{in } \Lambda\text{p} \\ &Bu_{i,n}\equiv \alpha(x_i, t_n)\,|x_i - \hat{x}_i|^{-1}[u(x_i,t_n) - u(\hat{x}_I, t_n)] + \beta_{i,n}u_{i,n}\ge 0 \quad \text{on } S_p \\ &u_{i,0}\ge 0 \quad \text{in } \Omega \end{aligned} \tag{2.6}$$

where $c_{i,n}\ge 0$; $d_{i,n}\ge 0$; $\hat{x}_i$ is a suitable point in $\Omega_p$ and $|x_i - \hat{x}_i|$ is the distance between $x_i$ and $\hat{x}_i$. $\alpha_{i,n}\ge 0$; $\beta_{i,n}\ge 0$; $\alpha_{i,n} + \beta_{i,n} > 0$; on $S_p$ and $c\equiv c_{i,n}$ is a bounded function in $\Lambda$p. Then $u_{i,n}\ge 0$ in $\bar{\Lambda}_p$. Moreover $u_{i,n} > 0$ in $\Lambda_p$, unless it is identically zero.

Proof is simple so details are omitted.

## 3. Monotone Scheme

**International Journal of Universal Mathematics and Mathematical Sciences**
ISSN: 2454-7271 Volume: 02, Issue: 01, Pages:01-13,
Published:June,2016 Web: www.universalprint.org ,
Title Key: Monotone Method For Finite Difference Equations…

Now, we develop monotone scheme for discrete time degenerate Dirichlet IBVP(2.3). We define upper and lower solutions of the time degenerate dicrete problem (2.3).

**Definition 3.1 :** A function $\tilde{u}_{i,n}$ in $\bar{\Lambda}_p$ is called upper solution of (2.3) if

$$\begin{aligned} & d_{i,n}k_n^{-1}\left(\tilde{u}_{i,n}-\tilde{u}_{i,n-1}\right)-L\left[\tilde{u}_{i,n}\right]\geq f_{i,n}\left(\tilde{u}_{i,n}\right); && (i,n)\in\Lambda_p \\ & \tilde{u}_{i,n}\geq g_{i,n}(\tilde{u}_{i,n}) && (i,n)\in S_p \\ & \tilde{u}_{i,0}\geq\psi_i, && i\in\Omega_p \end{aligned} \tag{3.1}$$

**Definition 3.2** : A function $\hat{u}_{i,n}$ in $\bar{\Lambda}_p$ is called lower solution of the discrete problem (2.3) if

$$\begin{aligned} & d_{i,n}k_n^{-1}\left(\hat{u}_{i,n}-\hat{u}_{i,n-1}\right)-L\left[\hat{u}_{i,n}\right]\leq f_{i,n}\left(\hat{u}_{i,n}\right); && (i,n)\in\Lambda_p \\ & \hat{u}_{i,n}\leq g_{i,n}(\hat{u}_{i,n}) && (i,n)\in S_p \\ & \hat{u}_{i,0}\leq\psi_i && i\in\Omega_p \end{aligned} \tag{3.2}$$

**Definition 3.3:** We denote the sector $S_{i,n}^{*}$ for any ordered upper and lower solutions $\tilde{u}_{i,n},\hat{u}_{i,n}$ and is defined as $S_{i,n}^{*}=\left\{u_{i,n}\in\bar{\Lambda}_p;\hat{u}_{i,n}\leq u_{i,n}\leq\tilde{u}_{i,n}\right\}$

**Definition 3.4** The functions $\tilde{u}_{i,n},\hat{u}_{i,n}$ are called ordered upper and lower solutions if $\tilde{u}_{i,n}\geq\hat{u}_{i,n}$ in $\bar{\Lambda}_p$.

**Monotone Iteration Scheme:** Consider the following iteration scheme with suitable initial iteration $u_{i,n}^{(0)}$,

$$\begin{aligned} & £[u_{i,n}^{(k)}]=F_{i,n}(u_{i,n}^{(k-1)}); && (i,n)\in\Lambda_p \\ & u_{i,n}^{(k)}+\underline{b}_{i,n}u_{i,n}^{(k)}=G_{i,n}(u_{i,n}^{(k-1)}); && (i,n)\in S_p \\ & u_{i,0}^{(k)}=\psi_i; && i\in\Omega_p \end{aligned} \tag{3.3}$$

For $k=1$ we have,

$$\begin{aligned} & £\left[u_{i,n}^{(1)}\right]=F_{i,n}\left(u_{i,n}^{(0)}\right) && (i,n)\in\Lambda_p \\ & u_{i,n}^{(1)}+\underline{b}_{i,n}u_{i,n}^{(1)}=G_{i,n}\left(u_{i,n}^{(0)}\right) && \text{on } S_p \\ & u_{i,0}^{(1)}(i,0)=\psi_i && \text{in } \Omega_p \end{aligned}$$

Since $u_{i,n}^{(0)}$ is known the R.H.S. is known. The existency theory for linear Parabolic IBVP implies that $u_{i,n}^{(2)}$ exists. Similarly for $k=2$ we have

$$\begin{aligned} & £\left[u_{i,n}^{(2)}\right]=F_{i,n}\left(u_{i,n}^{(1)}\right) && (i,n)\in\Lambda_p \\ & u_{i,n}^{(2)}+\underline{b}_{i,n}u_{i,n}^{(2)}=G_{i,n}\left(u_{i,n}^{(1)}\right) && \text{on } S_p \end{aligned} \tag{3.5}$$

**International Journal of Universal Mathematics and Mathematical Sciences**
ISSN: 2454-7271 Volume: 02, Issue: 01, Pages:01-13,
Published:June,2016 Web: www.universalprint.org ,
Title Key: Monotone Method For Finite Difference Equations…

$$u_{i,0}^{(2)} = \psi_i \qquad \text{in } \Omega_p$$

Since $u_{i,n}^{(1)}$ is known the R.H.S. is known. The existence theory for linear Parabolic IBVPs implies that $u_{i,n}^{(2)}$ exists. Thus for k = 3, 4, ……… we get $u_{i,n}^{(3)}, u_{i,n}^{(4)}, \ldots\ldots$

Thus we construct a sequence, the sequence is well defined follows from Lemma 2.1. We choose initial iteration $u_{i,n}^{(0)} = \tilde{u}_{i,n}$ and denote the sequence by $\overline{u}_{i,n}^{(k)}$. We also choose initial iteration $u_{i,n}^{(0)} = \tilde{u}_{i,n}$ and denote the sequence by $\underline{u}_{i,n}^{(k)}$. Thus choosing an upper solution or lower solution as the initial iterations, we get upper and lower sequences $\overline{u}_{i,n}^{(k)}$ and $\underline{u}_{i,n}^{(k)}$ respectively.

**Lemma 3.1 (Monotone Property) :** Suppose that

(i) $\tilde{u}_{i,n}, \hat{u}_{i,n}$ are ordered upper and lower solutions of nonlinear time degenerate Dirichlet IBVP (2.3),i.e.

$$d_{i,n} k_n^{-1}[u_{i,n} - u_{i,n-1}] - L[u_{i,n}] = f_{i,n}(u_{i,n}); \qquad \text{in } \Lambda_p$$
$$u_{i,n} = g_{i,n}(u_{i,n}); \qquad \text{on } S_p$$
$$u_{i,0} = \psi_i; \qquad i \in \Omega_p,$$

(ii) $f_{i,n}(u_{i,n})$ and $g_{i,n}(u_{i,n})$ satisfies the Lipschitz condition

$$-\underline{c}_{i,n}\left(u_{i,n}^{(1)} - u_{i,n}^{(2)}\right) \le f_{i,n}\left(u_{i,n}^{(1)}\right) - f_{i,n}\left(u_{i,n}^{(2)}\right) \le \overline{c}_{i,n}\left(u_{i,n}^{(1)} - u_{i,n}^{(2)}\right)$$
$$-\underline{b}_{i,n}\left(u_{i,n}^{(1)} - u_{i,n}^{(2)}\right) \le g_{i,n}\left(u_{i,n}^{(1)}\right) - g_{i,n}\left(u_{i,n}^{(2)}\right) \le \overline{b}_{i,n}\left(u_{i,n}^{(1)} - u_{i,n}^{(2)}\right) \qquad (3.6)$$

for $u_{i,n}^{(1)}, u_{i,n}^{(2)} \in <\hat{u}_{i,n}, \tilde{u}_{i,n}>$. Then the sequences $\overline{u}_{i,n}^{(k)}$ , $\underline{u}_{i,n}^{(k)}$ possess the monotone property.

$$\hat{u}_{i,n} \le \underline{u}_{i,n}^{(k)} \le \underline{u}_{i,n}^{(k+1)} \le \overline{u}_{i,n}^{(k+1)} \le \overline{u}_{i,n}^{(k)} \le \tilde{u}_{i,n} \qquad \text{in } \overline{\Lambda}_p. \qquad (3.7)$$

Moreover, $\overline{u}_{i,n}^{(k)}$ and $\underline{u}_{i,n}^{(k)}$ are ordered upper and lower solutions of time degenerate Dirichlet IBVP for k = 1, 2, 3, ……

**Proof:** Define

$$w_{i,n} = \overline{u}_{i,n}^{(0)} - \overline{u}_{i,n}^{(1)}$$
$$\therefore\ w_{i,n} = \tilde{u}_{i,n} - \overline{u}_{i,n}^{(1)}. \qquad \because u_{i,n}^{(0)} = \tilde{u}_{i,n}$$

Since $\tilde{u}_{i,n}$ is an upper solution, we have by definition 3.1

$$d_{i,n}k_n^{-1}\left[\tilde{u}_{i,n} - \tilde{u}_{i,n-1}\right] - L\left[\tilde{u}_{i,n}\right] \ge f_{i,n}\left(\tilde{u}_{i,n}\right) \ ; \text{ in } \Lambda p$$
$$\tilde{u}_{i,n} \ge g_{i,n}(\tilde{u}_{i,n}) \qquad i,n \in S_p$$
$$\tilde{u}_{i,0} \ge \psi_i, \qquad i \in \Omega_p,$$

**International Journal of Universal Mathematics and Mathematical Sciences**
ISSN: 2454-7271 Volume: 02, Issue: 01, Pages:01-13,
Published:June,2016 Web: www.universalprint.org ,
Title Key: Monotone Method For Finite Difference Equations…

Clearly

$$d_{i,n}k_n^{-1}[w_{i,n}-w_{i,n-1}]=d_{i,n}k_n^{-1}\left[\tilde{u}_{i,n}-\tilde{u}_{i,n-1}\right]-d_{i,n}k_m^{-1}\left[\bar{u}_{i,n}^{(1)}-\bar{u}_{i,n-1}^{(1)}\right]$$

$$-L[w_{i,n}]=-L\left[\tilde{u}_{i,n}\right]+L=\left[\bar{u}_{i,n}^{(1)}\right]$$

$$\underline{c}_{i,n}w_{i,n}=\underline{c}_{i,n}\tilde{u}_{i,n}-\underline{c}_{i,n}\bar{u}_{i,n}^{(1)}.$$

By adding we get

$$d_{i,n}k_n^{-1}\left[w_{i,n}-w_{i,n-1}\right]-L[w_{i,n}]+\underline{c}_{i,n}w_{i,n}\;=\left[d_{i,n}k_n^{-1}\;\tilde{u}_{i,n}-\tilde{u}_{i,n-1}\;-L[\tilde{u}_{i,n}]+\underline{c}_{i,n}\tilde{u}_{i,n}\right]$$

$$-\left[d_{i,n}k_n^{-1}\;\bar{u}_{i,n}^{(1)}-\bar{u}_{i,n-1}^{(1)}\;-L[\bar{u}_{i,n}^{(1)}]+\underline{c}_{i,n}\bar{u}_{i,n}^{(1)}\right]$$

$$=\left[d_{i,n}k_n^{-1}\;\tilde{u}_{i,n}-\tilde{u}_{i,n-1}\;-L[\tilde{u}_{i,n}]+\underline{c}_{i,n}\tilde{u}_{i,n}\right]-\left[\underline{c}_{i,n}\bar{u}_{i,n}^{(0)}+f_{i,n}\;\bar{u}_{i,n}^{(0)}\right]\quad \text{(By iterative scheme)}$$

$$=d_{i,n}k_n^{-1}\;\tilde{u}_{i,n}-\tilde{u}_{i,n-1}\;-L[\tilde{u}_{i,n}]+\underline{c}_{i,n}\tilde{u}_{i,n}-\underline{c}_{i,n}\tilde{u}_{i,n}-f_{i,n}(\tilde{u}_{i,n})$$

$$=d_{i,n}k_n^{-1}\;\tilde{u}_{i,n}-\tilde{u}_{i,n-1}\;-L[\tilde{u}_{i,n}]-f_{i,n}(\tilde{u}_{i,n})\geq 0$$

$$\therefore\; d_{i,n}k_n^{-1}[w_{i,n}-w_{i,n}]-L[w_{i,n}]+\underline{c}_{i,n}w_{i,n}\geq 0,\ \text{in } \Lambda_p$$

Also, $w_{i,n}+\underline{b}_{i,n}w_{i,n}.\geq 0$; on $S_p$ and

$$w_{i,0}=\tilde{u}_{i,0}-\bar{u}_{i,0}^{(1)}$$
$$=\tilde{u}_{i,0}-\Psi_i\geq 0$$

$\therefore\; w_{i,0}\geq 0$ in $\Omega$

Now applying the Lemma 2.1 we get, $w_{i,n}\geq 0$; in $\bar{\Lambda}_p$. This implies that

$$\bar{u}_{i,n}^{(1)}\leq \bar{u}_{i,n}^{(0)} \tag{3.8}$$

We also know that $\hat{u}_{i,n}$ is a lower solution.

Define $w_{i,n}=\bar{u}_{i,n}^{(1)}-\bar{u}_{i,n}^{(0)}$ and using $\underline{u}_{i,n}^{(0)}=\hat{u}_{i,n}$, We have $w_{i,n}=\underline{u}_{i,n}^{(1)}-\hat{u}_{i,n}$,

using the above argument we get, $\underline{u}_{i,n}^{(1)}\geq \underline{u}_{i,n}^{(0)}$. (3.9)

Next we define $w_{i,n}^{(1)}=\bar{u}_{i,n}^{(1)}-\underline{u}_{i,n}^{(1)}$

$$d_{i,n}k_n^{-1}\left[w_{i,n}^{(1)}-w_{i,n-1}^{(1)}\right]-L\left[w_{i,n}^{(1)}\right]+\underline{c}_{i,n}w_{i,n}^{(1)}$$

$$=\left[d_{i,n}k_n^{-1}\left[\bar{u}_{i,n}^{(1)}-\bar{u}_{i,n-1}^{(1)}\right]-L\left[\bar{u}_{i,n}^{(1)}\right]+\underline{c}_{i,n}\bar{u}_{i,n}^{(1)}\right]$$

$$-\left[d_{i,n}k_n^{-1}\left[\underline{u}_{i,n}^{(1)}-\underline{u}_{i,n-1}^{(1)}\right]-L\left[\bar{u}_{i,n}^{(1)}\right]+\underline{c}_{i,n}\underline{u}_{i,n}^{(1)}\right]$$

$=F_{i,n}\;\tilde{u}_{i,n}\;-f_{i,n}\;\hat{u}_{i,n}\;\geq 0.$ Also $w_{i,n}^{(1)}+\underline{b}_{i,n}w_{i,n}^{(1)}\geq 0$ ; $w_{i,0}^{(1)}=\bar{u}_{i,0}^{(1)}-\underline{u}_{i,0}^{(1)}=\Psi_i-\Psi_I\;=0$:

Applying the lemma 2.1 we get, $w_{i,0}^{(1)}\geq 0$. This shows that $\underline{u}_{i,0}^{(1)}\leq \bar{u}_{i,n}^{(1)}$ (3.10)

we conclude that $\underline{u}_{i,n}^{(0)}\leq \underline{u}_{i,n}^{(1)}\leq \bar{u}_{i,n}^{(1)}\leq \bar{u}_{i,n}^{(0)}$ (3.11)

Assume by induction $\underline{u}_{i,n}^{(k-1)}\leq \underline{u}_{i,n}^{(k)}\leq \bar{u}_{i,n}^{(k)}\leq \bar{u}_{i,n}^{(k-1)}$ in $\bar{\Lambda}_p$. (3.12)

**International Journal of Universal Mathematics and Mathematical Sciences**
ISSN: 2454-7271 Volume: 02, Issue: 01, Pages:01-13,
Published:June,2016 Web: www.universalprint.org ,
Title Key: Monotone Method For Finite Difference Equations…

Define function $w_{i,n}^{(k)} = \bar{u}_{i,n}^{(k)} - \bar{u}_{i,n}^{(k+1)}$.

Using iterative scheme and Lipschitz condition, $w^{(k)}$ satisfies the relation

$$\begin{aligned} d_{i,n}k_n^{-1}\left[w_{i,n}^{(k)} - w_{i,n-1}^{(k)}\right] - L\left[w_{i,n}^{(k)}\right] + \underline{c}_{i,n}w_{i,n}^{(k)} &= \left[d_{i,n}k_n^{-1}\left[\bar{u}_{i,n}^{(k)} - \bar{u}_{i,n-1}^{(k)}\right] - L\left[\bar{u}_{i,n}^{(k)}\right] + \underline{c}_{i,n}\bar{u}_{i,n}^{(k)}\right] \\ &\quad -\left[d_{i,n}k_n^{-1}\left[\bar{u}_{i,n}^{(k+1)} - \bar{u}_{i,n-1}^{(k+1)}\right] - L\left[\bar{u}_{i,n}^{(k+1)}\right] + \underline{c}_{i,n}\bar{u}_{i,n}^{(k+1)}\right] \\ &= \left[\underline{c}_{i,n}\bar{u}_{i,n}^{(k-1)} + f_{i,n}\ \bar{u}_{i,n}^{(k-1)}\right] - \left[\underline{c}_{i,n}\bar{u}_{i,n}^{(k)} + f_{i,n}\ \bar{u}_{i,n}^{(k)}\right] \\ &= F_{i,n}\ \bar{u}_{i,n}^{(k-1)} - F_{i,n}\ \bar{u}_{i,n}^{(k)} \geq 0 \end{aligned}$$

$\therefore d_{i,n}k_n^{-1}\left[w_{i,n}^{(k)} - w_{i,n-1}^{(k)}\right] - L\left[w_{i,n}^{(k)}\right] + \underline{c}_{i,n} \mid w_{i,n}^{(k)} \geq 0 \text{ in } \Lambda_p$ ; Also $w_{i,n}^{(k)} + \underline{b}_{i,n}w_{i,n}($

$= G_{i,n}\ \bar{u}_{i,n}^{(k-1)} - G_{i,n}\ \underline{u}_{i,n}^{(k)} \geq 0$

$w_{i,0}^{(0)} = \bar{u}_{i,0}^{(k)} - \bar{u}_{i,0}^{(k+1)} = \Psi_i - \Psi_i = 0 \quad \therefore w_{i,0}^{(k)} = 0, \text{ in } \Omega_p$

Again applying the lemma 2.1 we get, $w_{i,0}^{(k)} \geq 0$ ; i.e. $\bar{u}_{i,n}^{(k+1)} \leq \bar{u}_{i,n}^{(k)}$ (3.13)

Similarly consider $w_{i,n}^{(k)} = \underline{u}_{i,n}^{(k+1)} - \underline{u}_{i,n}^{(k)}$

$$\therefore d_{i,n}k_n^{-1}\left[w_{i,n}^{(k)} - w_{i,n-1}^{(k)}\right] - L\left[w_{i,n}^{(k)}\right] + \underline{c}_{i,n}w_{i,n}^{(k)} = \left[\underline{c}_{i,n}\underline{u}_{i,n}^{(k)} + f_{i,n}\ \underline{u}_{i,n}^{(k)}\right] - \left[\underline{c}_{i,n}\underline{u}_{i,n}^{(k-1)} + f_{i,n}\ \underline{u}_{i,n}^{(k-1)}\right]$$

(By using iterative scheme and Lipschitz condition),

$$\therefore d_{i,n}k_n^{-1}\left[w_{i,n}^{(k)} - w_{i,n-1}^{(k)}\right] - L\left[w_{i,n}^{(k)}\right] + \underline{c}_{i,n}w_{i,n}^{(k)} \geq 0 \text{ in } \Lambda_p$$

Also $w_{i,n}^{(k)} + \underline{b}_{i,n}w_{i,n}^{(k)} = G_{i,n}\ \underline{u}_{i,n}^{(k)} - G_{i,n}\ \underline{u}_{i,n}^{(k-1)} \geq 0$

$$\therefore w_{i,0}^{(k)} = 0, \text{ in } \Omega_p$$

Then we obtain $w_{i,0}^{(k)} \geq 0$. So $\underline{u}_{i,0}^{(k+1)} \geq \underline{u}_{i,0}^{(k)}$ (3.14)

Now define $w_{i,0}^{(k)} = \bar{u}_{i,n}^{(k+1)} - \underline{u}_{i,n}^{(k+1)}$

$$\begin{aligned} &d_{i,n}k_n^{-1}\left[w_{i,n}^{(k)} - w_{i,n-1}^{(k)}\right] - L\left[w_{i,n}^{(k)}\right] + \underline{c}_{i,n}w_{i,n}^{(k)} \\ &= F_{i,n}\ \bar{u}_{i,n}^{(k)} - F_{i,n}\ \underline{u}_{i,n}^{(k)} \geq 0 \\ &= F_{i,n}\ \underline{u}_{i,n}^{(k)} - F_{i,n}\ \underline{u}_{(i,n)}^{(k-0)} \geq 0 \end{aligned}$$

$$\therefore d_{i,n}k_n^{-1}\left[w_{i,n}^{(k)} - w_{i,n-1}^{(k)}\right] - L\left[w_{i,n}^{(k)}\right] + \underline{c}_{i,n}w_{i,n}^{(k)} \geq 0 \text{ in } \Lambda_p$$

Also $w_{i,n}^{(k)} + \underline{b}_{i,n}w_{i,n}^{(k)} \geq 0$

$$\therefore w_{i,0}^{(k)} = 0, \text{ in } \Omega_p$$

$w_{i,0}^{(k)} \geq 0$. So $\bar{u}_{i,0}^{(k+1)} \geq \underline{u}_{i,0}^{(k+1)}$ (3.15)

Thus we have, from the principle of induction

$$\hat{u}_{i,n} \leq \underline{u}_{i,n}^{(k)} \leq \underline{u}_{i,n}^{(k+1)} \leq \bar{u}_{i,n}^{(k+1)} \leq \bar{u}_{i,n}^{(k)} \leq \tilde{u}_{i,n} \text{ in } \bar{\Lambda}_p \text{ for every k = 1, 2, 3, ……}$$

**International Journal of Universal Mathematics and Mathematical Sciences**
ISSN: 2454-7271 Volume: 02, Issue: 01, Pages:01-13,
Published:June,2016 Web: www.universalprint.org ,
Title Key: Monotone Method For Finite Difference Equations…

This completes the proof.

**4. Applications:**

**Theorem 4.1 (Existence-Comparison Theorem)** Suppose that

(i) $\tilde{u}_{i,n}, \hat{u}_{i,n}$ are ordered upper and lower solutions of nonlinear time degenerate Neumann IBVP (2.3),

$$d_{i,n}k_n^{-1}\left[u_{i,n}-u_{i,n-1}\right]-L[u_{i,n}]=f_{i,n}(u_{i,n});\text{ in }\Lambda_p$$
$$u_{i,n}=g_{i,n}(u_{i,n});\quad \text{on } S_p$$
$$u_{i,0}=\psi_i;\quad i\in\Omega_p,$$

(ii) $f_{i,n}(u_{i,n})$ and $g_{i,n}(u_{i,n})$ satisfies the Lipschitz conditions

$$-\underline{c}_{i,n}\left(u_{i,n}^{(1)}-u_{i,n}^{(2)}\right)\le f_{i,n}\left(u_{i,n}^{(1)}\right)-f_{i,n}\left(u_{i,n}^{(2)}\right)\le-\bar{c}_{i,n}\left(u_{i,n}^{(1)}-u_{i,n}^{(2)}\right)$$
$$-\underline{b}_{i,n}\left(u_{i,n}^{(1)}-u_{i,n}^{(2)}\right)\le g_{i,n}\left(u_{i,n}^{(1)}\right)-g_{i,n}\left(u_{i,n}^{(2)}\right)\le-\bar{b}_{i,n}\left(u_{i,n}^{(1)}-u_{i,n}^{(2)}\right)\qquad\text{for } u_{i,n}^{(1)},u_{i,n}^{(2)}\in<\hat{u}_{i,n},\tilde{u}_{i,n}>.$$

Then the sequences $\left\{\bar{u}_{i,n}^{(k)}\right\}$, $\left\{\underline{u}_{i,n}^{(k)}\right\}$ converges monotonically to unique solution $u_{i,n}$ of time degenerate Dirichlet IBVP (2.3) and satisfy the relation

$$\hat{u}_{i,n}\le\underline{u}_{i,n}^{(1)}\le.....\le\underline{u}_{i,n}\le\bar{u}_{i,n}\le.....\le\bar{u}_{i,n}^{(1)}\le\tilde{u}_{i,n}\text{ in }\Lambda_p \tag{4.1}$$

**Proof:** We show the monotone convergence of the maximal and minimal sequences $\left\{\bar{u}_{i,n}^{(k)}\right\}$, $\left\{\underline{u}_{i,n}^{(k)}\right\}$ respectively. Suppose $w_{i,n}=\bar{u}_{i,n}^{(0)}-\bar{u}_{i,n}^{(1)}$ where $\bar{u}_{i,n}^{(0)}=\tilde{u}_{i,n}$. By using the definition 2.1 and iterative scheme, we have,

$$d_{i,n}k_n^{-1}(w_{i,n}-w_{i,n-1})-L[w_{i,n}]+\underline{c}_{i,n}w_{i,n}=\left[d_{i,n}k_n^{-1}(\tilde{u}_{i,n}-\tilde{u}_{i,n-1})-L[\tilde{u}_{i,n}]+\bar{c}_{i,n}\tilde{u}_{i,n}\right]$$
$$-\left[d_{i,n}k_n^{-1}(\bar{u}_{i,n}^{(1)}-\bar{u}_{i,n-1}^{(1)})-L\left[\bar{u}_{i,n}^{(1)}\right]+\bar{c}_{i,n}\bar{u}_{i,n}^{(1)}\right]$$
$$=\left[d_{i,n}k_n^{-1}(\tilde{u}_{i,n}-\tilde{u}_{i,n-1})-L\left[\tilde{u}_{i,n}\right]+\underline{c}_{i,n}\tilde{u}_{i,n}\right]-\left[c_{i,n}\bar{u}_{i,n}^{(0)}+f_{i,n}\left(\bar{u}_{i,n}^{(0)}\right)\right]$$
$$=d_{i,n}k_n^{-1}(\tilde{u}_{i,n}-\tilde{u}_{i,n-1})-L\left[\tilde{u}_{i,n}\right]+\underline{c}_{i,n}\tilde{u}_{i,n}-\underline{c}_{i,n}\tilde{u}_{i,n}-f_{i,n}(\tilde{u}_{i,n})=d_{i,n}k_n^{-1}(\tilde{u}_{i,n}-\tilde{u}_{i,n-1})-L\left[\tilde{u}_{i,n}\right]-f_{i,n}\left(\tilde{u}_{i,n}\right)\ge0$$
$$\therefore\ d_{i,n}k_n^{-1}(w_{i,n}-w_{i,n-1})-L[w_{i,n}]+\underline{c}_{i,n}w_{i,n}\ge0\text{ in }\Lambda_p$$

Also, $w_{i,n}+\underline{b}_{i,n}w_{i,n}.\ge0$ ; $w_{i,0}=\tilde{u}_{i,0}-\bar{u}_{i,0}^{(1)}=0$

$\therefore w_{i,0}\ge0,\ i\in\Omega_p$. By using lemma 2.1, we get, $w_{i,n}\ge0$ in $\bar{\Lambda}_p$.

$$\therefore\bar{u}_{i,n}^{(0)}\ge\bar{u}_{i,n}^{(1)} \tag{4.2}$$

We also know that $\hat{u}$ is a lower solution. Define $w_{i,n}=\underline{u}_{i,n}^{(1)}-\underline{u}_{i,n}^{(0)}$ and using $\underline{u}_{i,n}^{(0)}=\hat{u}_{i,n}$ $\therefore w_{i,n}=\underline{u}_{i,n}^{(1)}-\hat{u}_{i,n}$. Then we obtain, $w_{i,n}\ge0$. This gives $\underline{u}_{i,n}^{(1)}\ge\underline{u}_{i,n}^{(0)}$. (4.3)

Next suppose that $w_{i,n}^{(1)} = \bar{u}_{i,n}^{(1)} - \underline{u}_{i,n}^{(1)}$. Then by using the given Lipschitz condition, and by iterative scheme, we get,

$$d_{i,n}k_n^{-1}\ w_{i,n}^{(1)} - w_{i,n-1}^{(1)}\Big] - L\Big[w_{i,n}^{(1)}\Big] + \underline{c}_{i,n}w_{i,n}^{(1)} = \Big[\underline{c}_{i,n}\bar{u}_{i,n}^{(0)} + f_{i,n}\ \ \bar{u}_{i,n}^{(0)}\ \ \Big] - \Big[\underline{c}_{i,n}\underline{u}_{i,n}^{(0)} + f_{i,n}\ \ \underline{u}_{i,n}^{(0)}\ \ \Big]$$

$$= \Big[\underline{c}_{i,n}\tilde{u}_{i,n} + f_{i,n}\ \ \tilde{u}_{i,n}\ \ \Big] - \Big[\underline{c}_{i,n}\hat{u}_{i,n} + f_{i,n}\ \ \hat{u}_{i,n}\ \ \Big] = F_{i,n}\ \ \tilde{u}_{i,n}\ \ - F_{i,n}\ \ \hat{u}_{i,n}\ \ \geq 0 \text{ in } \Lambda_p$$

$$\therefore d_{i,n}k_n^{-1}\ \ w_{i,n}^{(1)} - w_{i,n-1}^{(1)}\ \ - L\Big[w_{i,n}^{(1)}\Big] + \underline{c}_{i,n}w_{i,n}^{(1)} \geq 0$$

Also $\quad w_{i,n}^{(k)} + \underline{b}_{i,n}w_{i,n}^{(k)} = G_{i,n}\ \ \bar{u}_{i,n}^{(0)}\ \ - G_{i,n}\ \ \underline{u}_{i,n}^{(0)}\ \ \geq 0$

$$\therefore\ w_{i,0}^{(1)} = 0,\ i \in \Omega_p.$$

Thus by Lemma 2.1, we get

$$w_{i,0}^{(1)} \geq 0.\quad \underline{u}_{i,0}^{(1)} \leq \bar{u}_{i,0}^{(1)} \tag{4.4}$$

Thus by (4.2), (4.3) and (4.4), we get

$$\underline{u}_{i,n}^{(0)} \leq \underline{u}_{i,n}^{(1)} \leq \bar{u}_{i,n}^{(1)} \leq \bar{u}_{i,n}^{(0)}.$$

Assume by induction

$$\underline{u}_{i,n}^{(k-1)} \leq \underline{u}_{i,n}^{(k)} \leq \bar{u}_{i,n}^{(k)} \leq \bar{u}_{i,n}^{(k-1)} \text{ in } \bar{\Lambda}_p$$

Define a function $w_{i,n}^{(k)} = \bar{u}_{i,n}^{(k)} - \bar{u}_{i,n}^{(k+1)}$

$$\therefore d_{i,n}k_n^{-1}\ \ w_{i,n}^{(k)} - w_{i,n-1}^{(k)}\Big] - L\Big[w_{i,n}^{(k)}\Big] + \underline{c}_{i,n}w_{i,n}^{(k)} = \Big[\underline{c}_{i,n}\bar{u}_{i,n}^{(k-1)} + f_{i,n}\ \ \bar{u}_{i,n}^{(k-1)}\ \ \Big] - \Big[\underline{c}_{i,n}\bar{u}_{i,n}^{(k)} + f_{i,n}\ \ \bar{u}_{i,n}^{(k)}\ \ \Big]$$

(By using definition 2.1 and iterative scheme)

$$= F_{i,n}\ \ \bar{u}_{i,n}^{(k-1)}\ \ - F_{i,n}\ \ \bar{u}_{i,n}^{(k)}\ \ \geq 0$$

(By using Lipschtiz condition)

$$\therefore d_{i,n}k_n^{-1}\ \ w_{i,n}^{(k)} - w_{i,n-1}^{(k)}\ \ - L\Big[w_{i,n}^{(k)}\Big] + \underline{c}_{i,n}w_{i,n}^{(k)} \geq 0;\ \Lambda_p.$$

Also, $w_{i,n}^{(k)} + \underline{b}_{i,n}w_{i,n}^{(k)} = G_{i,n}\ \ \bar{u}_{i,n}^{(k-1)}\ \ - G_{i,n}\ \ \bar{u}_{i,n}^{(k)}\ \ \geq 0$

$\therefore\ w_{i,0}^{(k)} = 0,\ i \in \Omega_p$. Then we obtain

$$w_{i,0}^{(k)} \geq 0. \text{ so } \bar{u}_{i,n}^{k} \geq \bar{u}_{i,n}^{(k+1)} \tag{4.5}$$

Now define consider $w_{i,n}^{(k)} = \underline{u}_{i,n}^{(k+1)} - \underline{u}_{i,n}^{(k)}$;

$$\therefore d_{i,n}k_n^{-1}\Big[w_{i,n}^{(k)} - w_{i,n-1}^{(k)}\Big] - L\Big[w_{i,n}^{(k)}\Big] + \underline{c}_{i,n}w_{i,n}^{(k)}$$

$$= \Big[\underline{c}_{i,n}u_{i,n}^{(k)} + f_{i,n}\ \ \underline{u}_{i,n}^{(k)}\ \ \Big] - \Big[\underline{c}_{i,n}\underline{u}_{i,n}^{(k-1)} + f_{i,n}\ \ \underline{u}_{i,n}^{(k-1)}\ \ \Big]$$

(By using iterative scheme and Lipschitz condition),

$$\therefore d_{i,n}k_n^{-1}\Big[w_{i,n}^{(k)} - w_{i,n-1}^{(k)}\Big] - L\Big[w_{i,n}^{(k)}\Big] + \underline{c}_{i,n}w_{i,n}^{(k)} \geq 0 \text{ in } \Lambda_p$$

Also $\quad w_{i,n}^{(k)} + \underline{b}_{i,n}w_{i,n}^{(k)} = G_{i,n}\ \ \underline{u}_{i,n}^{(k)}\ \ - G_{i,n}\ \ \underline{u}_{i,n}^{(k-1)}\ \ \geq 0$ ; $\therefore w_{i,0}^{(k)} = 0$, in $\Omega_p$

Then we obtain $w_{i,n}^{(k)} \geq 0$. So $\underline{u}_{i,n}^{(k+1)} \geq \underline{u}_{i,n}^{(k)}$ (4.6)

$$\therefore d_{i,n}k_n^{-1}\left(w_{i,n}^{(k)} - w_{i,n-1}^{(k)}\right) - L\left[w_{i,n}^{(k)}\right] + \underline{c}_{i,n}w_{i,n}^{(k)} = \left[\underline{c}_{i,n}\underline{u}_n^{(k)} + f_{i,n}\left(\underline{u}_n^{(k)}\right)\right] - \left[\underline{c}_{i,n}\underline{u}_n^{(k-1)} + f_{i,n}\left(\underline{u}_n^{(k-1)}\right)\right]$$

(By definition 2.1 and using iterative scheme)

$$= F_{i,n}\left(\bar{u}_{i,n}^{(k)}\right) - F_{i,n}\left(\underline{u}_{i,n}^{(k)}\right) \geq 0$$

(By using Lipschtiz condition)

$$\therefore d_{i,n}k_n^{-1}\left(w_{i,n}^{(k)} - w_{i,n-1}^{(k)}\right) - L\left[w_{i,n}^{(k)}\right] + \underline{c}_{i,n}w_{i,n}^{(k)} \geq 0;\ \Lambda_p.$$

Also, $w_{i,n}^{(k)} + \underline{b}_{i,n}w_{i,n}^{(k)} = G_{i,n}\left(\bar{u}_{i,n}^{(k)}\right) - G_{i,n}\left(\underline{u}_{i,n}^{(k)}\right) \geq 0$ ; $\therefore\ w_{i,0}^{(k)} = 0,\ i \in \Omega_p$.

$$w_{i,n}^{(k)} \geq 0.;\quad \text{so } \bar{u}_{i,n}^{(k+1)} \geq \underline{u}_{i,n}^{(k+1)} \tag{4.7}$$

Thus from (4.5), (4.6) and (4.7) we have, $\underline{u}_{i,n}^{(k)} \leq \underline{u}_{i,n}^{(k+1)} \leq \bar{u}_{i,n}^{(k+1)} \leq \bar{u}_{i,n}^{(k)}$. (4.8)

Thus monotone property (3.7) follows from principle of mathematical induction. Now we conclude that the sequence $\left\{\bar{u}_{i,n}^{(k)}\right\}$ is a monotone nonincreasing and is bounded from below hence it is convergent. Also the sequence $\left\{\underline{u}_{i,n}^{(k)}\right\}$ is monotone nondecreasing and is bounded from above. Hence it is convergent. So, $\lim_{k\to\infty}\bar{u}_{i,n}^{(k)} = \bar{u}_{i,n}$ and $\lim_{k\to\infty}\underline{u}_{i,n}^{(k)} = \underline{u}_{i,n}$ exists and called maximal and minimal solutions respectively of the time degenerate parabolic Dirichlet initial boundary value problem (2.3) and they satisfy $\hat{u}_{i,n} \leq \underline{u}_{i,n}^{(1)} \leq \underline{u}_{i,n}^{(2)} \leq \ldots\ldots \leq \underline{u}_{i,n} \leq \bar{u}_{i,n} \leq \ldots\ldots \leq \bar{u}_{i,n}^{(2)} \leq \bar{u}_{i,n}^{(1)} \leq \tilde{u}_{i,n}$. This completes the proof.

**Theorem 4.2 [Uniqueness Theorem]** Suppose that

(i) $\tilde{u}_{i,n}, \hat{u}_{i,n}$ are ordered upper and lower solutions of time degenerate Dirichlet IBVP (2.3) i.e.

$$d_{i,n}k_n^{-1}\left[u_{i,n} - u_{i,n-1}\right] - L\left[u_{i,n}\right] = f_{i,n}(u_{i,n});(i,n) \in \Lambda_p$$
$$u_{i,n} = g_{i,n}(u_{i,n});\quad \text{on } S_p$$
$$u_{i,0} = \psi_i;\quad i \in \Omega_p, \text{ with } \tilde{u} \geq \hat{u}_{i,n},$$

(ii) The functions $f_{i,n}(u_{i,n})$ and $g_{i,n}(u_{i,n})$ satisfies the Lipschtiz condition

$$-\underline{c}_{i,n}\left(u_{i,n}^{(1)} - u_{i,n}^{(2)}\right) \leq f_{i,n}\left(u_{i,n}^{(1)}\right) - f_{i,n}\left(u_{i,n}^{(2)}\right) \leq \bar{c}_{i,n}\left(u_{i,n}^{(1)} - u_{i,n}^{(2)}\right)$$

$$-\underline{b}_{i,n}\left(u_{i,n}^{(1)} - u_{i,n}^{(2)}\right) = g_{i,n}\left(u_{i,n}^{(1)}\right) - g_{i,n}\left(u_{i,n}^{(2)}\right) \leq \bar{b}_{i,n}\left(u_{i,n}^{(1)} - u_{i,n}^{(2)}\right) \quad \text{for } \hat{u}_{i,n} \leq u_{i,n}^{(2)} \leq u_{i,n}^{(1)} \leq \tilde{u}_{i,n}. \text{ holds}$$

$$d_{i,n}k_n^{-1} > \underline{c}_{i,n} - \underline{b}_{i,n}.$$

Then the discrete problem (2.3) has a unique solution.

**Proof :** We know that $\bar{u}_{i,n}, \underline{u}_{i,n}$ are maximal and minimal solutions respectively of the discrete time degenerate Dirichlet IBVP (2.3). To prove uniqueness we suppose that,

$$w_{i,n} = \bar{u}_{i,n} - \underline{u}_{i,n}\ d_{i,n}k_n^{-1}(w_{i,n} - w_{i,n-1}) = L\left[w_{i,n}\right] = f_{i,n}(w_{i,n}) = f_{i,n}\left(\bar{u}_{i,n}\right) - f_{i,n}\left(\underline{u}_{i,n}\right)$$

$$\geq -\underline{c}_{i,n}\left(\bar{u}_{i,n}-\underline{u}_{i,n}\right)$$

$$\geq -\underline{c}_{i,n}w_{i,n}.\therefore d_{i,n}k_n^{-1}(w_{i,n}-w_{i,n-1})-L[w_{i,n}]+\underline{c}_{i,n}w_{i,n}\geq 0 \text{ in } \Lambda_p$$

$$|x_i-\hat{x}_i|^{-1}\left[w(x_i,t_n)-w\left(\hat{x}_i,t_n\right)\right]+\underline{b}_{i,n}w_{i,n}$$

$$=|x_i-\hat{x}_i|^{-1}\left[\bar{u}(x_i,t_n)-\bar{u}\left(\hat{x}_i,t_n\right)\right]-|x_i-\hat{x}_i|^{-1}\left[\underline{u}(x_i,t_n)-\underline{u}\left(\hat{x}_i,t_n\right)\right]+\underline{b}_{i,n}\bar{u}_{i,n}-\underline{b}_{i,n}\underline{u}_{i,n}$$

$$=\left[|x_i-\hat{x}_i|^{-1}\left[\bar{u}(x_i,t_n)-\bar{u}\left(\hat{x}_i,t_n\right)\right]+\underline{b}_{i,n}\bar{u}_{i,n}\right]-\left[|x_i-\hat{x}_i|^{-1}\left[\underline{u}(x_i,t_n)-\underline{u}\left(\hat{x}_i,t_n\right)\right]+\underline{b}_{i,n}\bar{u}_{i,n}\right]$$

$$=\left[g_{i,n}\left(\bar{u}_{i,n}\right)-\underline{b}_{i,n}\left(\bar{u}_{i,n}\right)\right]-\left[g_{i,n}\left(\underline{u}_{i,n}\right)-\underline{b}_{i,n}\left(\underline{u}_{i,n}\right)\right]$$

(by using iterative scheme),

$$=G_{i,n}\left(\bar{u}_{i,n}\right)-G_{i,n}\left(\underline{u}_{i,n}\right)\geq 0; \text{ on } S_p$$

$$w_{i,n}=\bar{u}_{i,n}-\underline{u}_{i,n}=\Psi_i-\Psi_i=0 \quad \text{in } S_p$$

By using the positivity lemma we get $w_{i,n}\geq 0$ in $\bar{\Lambda}_p$

$$\therefore\ \bar{u}_{i,n}\geq \underline{u}_{i,n} \tag{4.9}$$

Also we can show that $\bar{u}_{i,n}\leq \underline{u}_{i,n}$ (4.10)

Inequalities (4.9) and (4.10) implies that $\bar{u}_{i,n}\equiv \underline{u}_{i,n}$ in $\Lambda_p$.∴ IBVP (2.3) has unique solution.

Hence the result.

**5 .Results:** Extended the well known method of upper –lower solutions for continuous parabolic problem to finite difference system of nonlinear time degenerate parabolic Dirichlet initial boundary value problem.

**6.Discussion:** Researchers can be extend the well known method of upper –lower solutions for continuous nonlinear parabolic problem to finite difference system of nonlinear time degenerate parabolic Mixed initial boundary value problem .

**7.Acknowledgement :** I am thankful to Prof. D.B.
Dhaigude ,Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad for helpful discussions. We are also grateful to the referee for his valuable suggestions.

**References:**

[1] Ames, W. F., *Numerical Methods for Partial Differential Equations*, (Third edition), Academic Press, San Diego, 1992.

[2] Dhaigude, D. B., *Sturmain Theorems for Time Degenerate Parabolic Inequalities*. A. St.Univ. *"Al. I. Cza"* Iasi Mathematics, (1990). 36: 35-39.

[3] Dhaigude, D. B. and Kasture, D. Y., *Nonlinear Time Degenerate Parabolic Inequalities and Application.* J. Math. Anal. and Appl. (1993)., 175(2): 476-481.

[4] Dhaigude R. M., *Monotone Method for Nonlinear Weakly Coupled Time Degeneratre Parabolic System and Applications*. Vidarbha Journal of Science (2012)., 7(1-2): 33-49.

[5] Forsythe, G. E. and Wasow, *W. R., Finite Difference Methods for Partial Differentia Equations*, Weley, New York, 1964.

[6] Ippolite, P. M, *Maximum Principles and Classical Solutions for Degenerate Parabolic Equations*, J. Math. Anal., 64 (1978), 530{561

[7] Ladde, G. S., *Lakshamikantham, V. and Vatsala, A.S., Monotone Iterative Techniques for Nonlinear Digfferential Equations*, Pitman, New York, 1985.

[8] Pao, C. V., *Monotone iterative Method of Finite Difference System of Reaction Diffusion Equations, Numer*. Math., 46 (1985), 571-586.

[9] Pao, C. V., *Nonlinear Parabolic and Elliptic Equation*, Plenum Press, New York, 1992.

[10] Sattinger, D. H., *Monotone Methods in Nonlinear Elliptic and Parabolic Boundary Value Problems*, Indiana Uni., Math. J., 21 (1972), 979-1000