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Study of Fractional Transforms and Its Applications

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Abstract

The aim of this paper is to study the fractional Mellin and fractional Fourier transforms. As an application of these fractional transform, we had proven some properties and solvedfractional order differential equations.

Keywords: Gamma function, fractional Mellin Transform, fractional Fourier Transform,
fractional derivative

Subject Classification: Subject Classification 2015.

1. Introduction:

The applications of fractional transforms to generalized function have been done timeto time and their properties have been studied by various mathematicians. As Fouriertransform has shift invariant property and scale invariant property of Mellin transform,the Fourier-Mellin transform is a very powerful tool for model problems in signalprocessing and other applications. Furthermore, the fractional Fourier transform wasproposed by Namias and developed by McBrideit .After that,it has been studied bymany researchers and contributed. Also, L.B.Almeida had introduced the fractionalFourier transform as an angular transform. The fractional calculus have several applicationsin various fields of Mathematics as well as in real life situations, such asAbel's integral equation, viscoelasticity, capacitor theory, conductance of biologicalsystems [1, 6, 8]. The idea of fractional operators, fractional derivative, fractionalgeometry has long back history but fractional transform has been rediscovered inquantum mechanics, optics, signal processing as well as in pattern recognition.

Now a days, many linear boundary value and initial value problems in appliedmathematics, mathematical physics, and engineering science are effectively solved byfractional Fourier and fractional Mellin transforms. Also, these transforms are usefulfor fractional integral equations. To solve these fractional differential equations, weneed various fractional transformations like Fractional Laplace Transform, FractionalFourier Transform [7], Fractional Mellin Transform [2, 3, 4, 8], Fractional WaveletTransform, Fractional Hankel Transform etc.Therefore, this is a field of active researchand it is still in full expansion.

The paper is organized as follows:

In section 2, we have given some basic definitions which are useful for the further calculations. In section 3, we had proved some properties of fractional Transformations. In last section as an application we have solved some fractional differential equations by using these Transformations along with the conclusion.

2. Preliminaries

2.1) Grūunwald – Letnikov:

The Grūunwald – Letnikov definition of fractional derivative of a function generalize the notion of backward difference quotient of integer order. The Grūunwald – Letnikov fractional derivative of order α of the function f(x) is define as [8]

$$ {}_aD_x^{\alpha} f(x) = \lim_{N\to\infty}\{\frac{(\frac{x-a}{N})^{-\alpha}}{\Gamma(-\alpha)}\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha)}{\Gamma(j+1)}f(x-j[\frac{x-a}{N}])\} $$

In this case α = 1 if the limit exists the Grūunwald – Letnikov fractional derivative is the left derivative of the function.

2.2) (Riemann-Liouville): Riemann (1953), Liouville (1832)

(a) If *f(x)* ∈C [a, b] and $a < x < b$then

$$ I_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}dt $$

Where, α ∈ (−∞,∞) is called the Riemann-Liouville fractional integral of order α.

(b) If *f(x)* ∈C [a, b] and $a < x < b$then

$${}_aD_x^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^{\alpha}} dt \quad 0 < \alpha < 1$$

is called the Riemann-Liouville fractional derivative of order α [8]

2.3) <u>(M. Caputo (1967)):</u>

If f(x) ∈C[a, b] and a < x < b then the Caputo fractional derivative of order α is defined as follows [6],

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{f'(t)}{(x-t)^{\alpha}} dt \quad 0 < \alpha < 1$$

2.4) <u>Fractional Fourier Transform (FrFT)</u>

The Fractional Fourier transform of function *f(t)* is defined as follows [5],

$$\widehat{f_\alpha}(\xi) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \xi^{\frac{1}{\alpha}}} dt \ , 0 < \alpha \leq 1$$

Where, as α → 1, the FrFT tends to be an ordinary Fourier Transform, it is defined forthe function space called Lizorkin space. This space is denoted by Φ(ℜ). Also, theabove definition is applicable for some rapidly decreasing functions. If the functionsu(x) and v(x) are in Lizorkin space then we have the following property.

2.5) <u>Inverse Fractional Fourier Transform</u>

The Fractional Fourier transform of function $\widehat{f_\alpha}(\xi)$ is defined as follows [5],

$$f(t) = \int_{-\infty}^{\infty} \widehat{f_\alpha}(\xi) e^{2\pi i \xi^{\frac{1}{\alpha}}} dt \ , 0 < \alpha \leq 1$$

2.6) <u>Fractional Mellin Transform (FrMT)</u>

The Fractional Mellin transform of the function f(t) with angular parameter ϕ is denoted by M^{Φ}(f(x)) and it is defined as follows [2],

$[M^{\Phi}(f(x))](s) = \{\sqrt{1 - i\cot\Phi} \int_0^{\infty} \frac{f(x)}{\sqrt{x}} e^{i\pi\varphi(x)} dx \ for \ \Phi \neq n\pi$, where $\varphi(x)$is given by$\varphi(x) = s^2 \cot\Phi + (\ln x)^2 \cot\Phi - 2s(\ln x)\csc\Phi$

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2.7) Inverse Fractional Mellin Transform

The Fractional Mellin transform of the function is denoted by $\boldsymbol{M^{-\Phi}}$(f(x)) and it is defined as follows [2],

$\boldsymbol{M^{-\Phi}}$(f(x)) = $\frac{\boldsymbol{1}}{\boldsymbol{2\pi i}}\int_{\boldsymbol{c-i\infty}}^{\boldsymbol{c+i\infty}}\boldsymbol{x^{-y}}\emptyset(\boldsymbol{y})\boldsymbol{dy}$, the transform $\emptyset(\boldsymbol{y})$ exist if the integral $\int_{\boldsymbol{0}}^{\infty}|\boldsymbol{f(x)}|\boldsymbol{x^{k-1}dx}$ is bounded for some k > 0

2.8) Convolution: The convolution of functions f(τ) and g(τ) at point t is defined as,

(f∗ g)(t) = $\int_{-\infty}^{\infty}\boldsymbol{f(\tau)g(t-\tau)d\tau}$

3. Properties of fractional transforms

(a) Properties of Fractional Fourier Transform

I) If g(t) = f(−t) then,

$$\widehat{g_\alpha}(\xi) = \int_{-\infty}^{\infty} g(t)e^{-2\pi i\xi^{\frac{1}{\alpha}}}dt$$

$$=\int_{-\infty}^{\infty} f(-t)e^{-2\pi i\xi^{\frac{1}{\alpha}}}dt \qquad (3.1)$$

On substituting −t = x in equation (3.1) ,
we get

$$\widehat{g_\alpha}(\xi) = \widehat{f_\alpha}(-\xi)$$

II) If $g(t) = f(t+\beta)$ where β ∈ℜ then

$$\widehat{g_\alpha}(\xi) = \int_{-\infty}^{\infty} g(t)e^{-2\pi i\xi^{\frac{1}{\alpha}}}dt$$

$$=\int_{-\infty}^{\infty} f(-t)e^{-2\pi i\xi^{\frac{1}{\alpha}}}dt \qquad (3.2)$$

On substituting $t + \beta = x$ in equation (3.2) , we get

$$\widehat{g_\alpha}(\xi) = e^{-2\pi i\beta\xi^{\frac{1}{\alpha}}}\widehat{f_\alpha}(\xi)$$

III) If $g(t) = f(\beta t)$, where β ≠ 0 then

$$\widehat{g_\alpha}(\xi) = \int_{-\infty}^{\infty} g(t)e^{-2\pi i\xi^{\frac{1}{\alpha}}}dt$$

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$$=\int_{-\infty}^{\infty} f(\beta t)e^{-2\pi i\xi^{\frac{1}{\alpha}}}dt \qquad (3.3)$$

On substituting βt = x equation (3.3) , we get

$$\widehat{g_\alpha}(\xi) = \frac{1}{\beta}\widehat{f_\alpha}\left(\frac{\xi}{\beta^\alpha}\right)$$

IV) If *h(t)* = *(f* ∗ *g)(t)*, then

$$\widehat{h_\alpha}(\xi) = \int_{-\infty}^{\infty} h(t)e^{-2\pi i\xi^{\frac{1}{\alpha}}}dt$$

$$=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau\right] e^{-2\pi i\xi^{\frac{1}{\alpha}}}dt$$

from definition (2.8) by applying fubini's theorem to the above equation we have,

$$\widehat{h_\alpha}(\xi) = \widehat{f_\alpha}(\xi)\widehat{g_\alpha}(\xi)$$

(b) Properties of Fractional Mellin Transform

I) If *g(x)*= *f(kx), k > 0* then by definition (2.6)

$$[M^\Phi(\mathrm{g(x)})](\mathrm{s}) = \sqrt{1-icot\Phi}\int_0^\infty \frac{f(kx)}{\sqrt{x}}e^{i\pi\varphi(x)}dx \qquad (3.4)$$

and $\boldsymbol{\varphi(x) = s^2 cot\Phi + (\ln x)^2 \cot\Phi - 2s(\ln x)\csc\Phi}$

bysubstituting*kx* =*t* in equation (3.4), we get

$$[M^\Phi(\mathrm{f(kx)})](\mathrm{s}) = \varphi(\Phi,\mathrm{s},\mathrm{k})[M^\Phi e^{-2\pi i(\ln t)(\ln k)\cot\Phi}(\mathrm{f(t)})](\mathrm{s})$$

Where, $\varphi(\Phi,\mathrm{s},\mathrm{k}) = \sqrt{1-icot\Phi}\frac{1}{\sqrt{k}}e^{2\pi is(\ln k)\csc\Phi + i\pi(\ln k)^2 cot\Phi}$

II) If g(x) = x $^{\alpha}$ f(x), then

$$[M^\Phi(\mathrm{g(x)})](\mathrm{s}) = \sqrt{1-icot\Phi}\int_0^\infty \frac{x^\alpha f(x)}{\sqrt{x}}e^{i\pi\varphi(x)}dx \qquad (3.5)$$

$$\boldsymbol{\varphi(x) = s^2 cot\Phi + (\ln x)^2 \cot\Phi - 2s(\ln x)\csc\Phi}$$

by rearranging the terms in equation (3.5),

$$[\boldsymbol{M^\Phi}(\mathrm{g(x)})](\mathrm{s}) = \boldsymbol{\sqrt{1-icot\Phi}e^{i\pi s^2\cot\Phi}\int_0^\infty \frac{f(x)}{\sqrt{x}}e^{\varphi(x)}dx}$$

Where
$\varphi(x) = i\pi[(ln\, x)^2 cot\, \Phi - 2(s+\alpha) \ln x \csc \Phi]$
$+ \alpha ln\, x[1 + 2\pi i\, csc\Phi] - i\pi[s^2 + (s+\alpha)^2]$

III) If $g(x) = f(x+a)$

$[M^{\Phi}(g(x))](s) = \sqrt{1 - icot\Phi} \int_0^\infty \frac{f(\alpha+x)}{\sqrt{x}} e^{i\pi\varphi(x)} dx$ (3.6)

and

$\varphi(x) = s^2 cot\Phi + (\ln x)^2 \cot \Phi - 2s(\ln x) \csc \Phi$

with acuteangle in equation (3.6),

$[M^{\Phi}(f(x+\alpha)](s) = \int_\alpha^\infty \frac{f(t)}{\sqrt{(t-\alpha)}} e^{i\pi(-2s[\ln(t-\alpha)])} dt$

$= \int_\alpha^\infty \frac{f(t)}{(t-\alpha)^{u-1}} dt$

Where

u = (½) – $2si\pi$

In the next section we solve some fractional order differential equations using fractional transforms and their solutions are given.

4. Applications of fractional Transforms

Example I:

Consider the following fractional order differential equation,

${}_0D_x^\alpha y(x) = e^{-x}, 0 < \alpha < 1$ (4.1)

with initial condition y(0)=0

To solve the above equation by using the definition (2.2 a), we have

$$\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{y(t)}{(x-t)^\alpha} dt = e^{-x}$$

By applying FrMT to both sides of the above equation with parameter$\Phi = \frac{\pi}{2}$, we have

$[M^{\frac{\pi}{2}}({}_0D_x^\alpha y(x))](s) = M^{\frac{\pi}{2}}(e^{-x})(s)$

$\sqrt{1 - icot\Phi} \int_0^\infty \frac{D_x^\alpha y(x)}{\sqrt{x}} e^{i\pi\varphi(x)} dx = \Gamma(u)$

Where u = (½) – $2is\pi$

$$\Rightarrow \frac{\Gamma(1-u+\alpha)Y(u-\alpha)}{\Gamma(1-\text{u})} = \Gamma(u)$$

$$\Rightarrow Y(u-\alpha) = \frac{\Gamma(u)\Gamma(1-u)}{\Gamma(1-u+\alpha)}$$

$\Rightarrow Y(u-\alpha) = \beta(1-u,\alpha)\frac{\Gamma(u)}{\Gamma(\alpha)}$ (A)

Define $G(u) = \frac{\Gamma(u)}{\Gamma(u+\alpha)}, F(u) = \frac{\Gamma(u)}{\Gamma(\alpha)}$

Applying inverse fractional Complex Mellin transform on both sides to the equation (A)

$$t^{\alpha} y(x) = M^{-\frac{\pi}{2}}(G(1-u)F(u))$$

Now we define function as follows

$$g(t) = \begin{cases} \dfrac{(1-t)^{\alpha-1}}{\Gamma(\alpha)} & ,0<t<1 \\ 0, & otherwise \end{cases}$$

By using Convolution theorem for Complex FrMT we get the solution as follows:

$$y(x) = \frac{x^{\alpha}}{\Gamma(\alpha)}\int_0^1 e^{-x\xi}\, x^{-\alpha}\frac{1}{(x-\xi)^{1-\alpha}}d\xi$$

which is the required solution of the given fractional differential equation

Example II:

Consider the following fractional order differential equation,

$-\infty D_x^{\alpha} y(x) = f(t), 0 < \alpha \leq 1$ (4.2)

with initial condition y(0)=0

$$f(t) = \begin{cases} e^{-t}, 0 \leq t < \infty \\ 0, \ otherwise \end{cases}$$

Apply FrFT on both sides to the equation (4.2)

$$\frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^{\infty}\left[\frac{d}{dt}\int_{-\infty}^{0}(t-\tau)^{-\alpha}y(\tau)d\tau\right]e^{-2\pi it\xi^{\frac{1}{\beta}}}dt = \int_0^{\infty} e^{-(1+2\pi it\xi^{\frac{1}{\beta}})}dt$$

$$\Rightarrow \frac{2\pi i\xi^{\frac{1}{\beta}}}{\Gamma(1-\alpha)}\int_{-\infty}^{\infty}\left[\int_0^{\infty}(t-\tau)^{-\alpha}y(\tau)d\tau\right]e^{-2\pi it\xi^{\frac{1}{\beta}}}dt = \int_0^{\infty} e^{-(1+2\pi it\xi^{\frac{1}{\beta}})}dt$$

we define the function as follows,

$h(t-\tau) = \begin{cases} \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}, 0<\tau<t \\ 0, otherwise \end{cases}$ and

$$H(\tau) = \begin{cases} 0, \tau < 0 \\ 1, \tau > 0 \end{cases}$$

Now by convolution theorem (2.8)

$$\int_{-\infty}^{\infty} [(H(\tau)h(\tau)] * y(\tau)(t)e^{-2\pi it\xi^{\frac{1}{\beta}}}dt = \frac{(2\pi i\xi^{\frac{1}{\beta}})^{-1}}{1+2\pi i\xi^{\frac{1}{\beta}}}$$

$$\Rightarrow \widehat{y_\beta}(\xi)\int_0^t t^{-\alpha}e^{-2\pi it\xi^{\frac{1}{\beta}}}dt = \frac{(2\pi i\xi^{\frac{1}{\beta}})^{-1}}{1+2\pi i\xi^{\frac{1}{\beta}}}$$

$$\Rightarrow \left(2\pi i\xi^{\frac{1}{\beta}}\right)^{\alpha-1}\widehat{y_\beta}(\xi) = \frac{(2\pi i\xi^{\frac{1}{\beta}})^{-1}}{1+2\pi i\xi^{\frac{1}{\beta}}}$$

$\Rightarrow \widehat{y_\beta}(\xi) = \frac{(2\pi i\xi^{\frac{1}{\beta}})^{-\alpha}}{1+2\pi i\xi^{\frac{1}{\beta}}}$ --- I

Define, $h(t-\tau) = \begin{cases} \frac{(t-\tau)^{-(1-\alpha)}}{\Gamma(\alpha)}, 0 < \tau < t \\ 0, otherwise \end{cases}$ (A)

$g(\tau) = \begin{cases} e^{-\tau}, 0 \le \tau < \infty \\ 0, otherwise \end{cases}$ (B)

Now, $\widehat{h_\beta}(\xi) = (2\pi i\xi^{\frac{1}{\beta}})^{-\alpha}$ and $\widehat{g_\beta}(\xi) = \frac{1}{1+2\pi i\xi^{\frac{1}{\beta}}}$

Hence from equation I, (A) and (B)

$$\widehat{y_\beta}(\xi) = \widehat{h_\beta}(\xi)\widehat{g_\beta}(\xi)$$

Applying inverse FrFT on both sides to the equation and using the definitionof convolution

We get,

$$\boldsymbol{y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\, e^{-\tau}d\tau}$$

which is the required solution of the given fractional differential equation.

5. Conclusion

The aim of this manuscript is to solve the fractional differential equation by using fractional transformation along with it we have proved some properties of it.

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