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On Some General Integral Inequalities

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Abstract:-
Abstract. In this paper, we establish some new integral inequalities by analytic approach.

Key words: Cauchy Inequality, Integral inequality

1. Introduction:

Most of the inequalities developed so far in the literature perform quite well in practice and hence have found widespread acceptance in a variety of applications. Because of this, it is not surprising that numerous studies of new types of inequalities have been made in order to achieve many new developments in various branches of mathematics. By the desire to widen the scope of such inequalities, recently many papers have appeared which deal with large number of inequalities applicable in situations in which the earlier inequalities do not apply directly. During the past years so many authors have developed various integral inequalities [4, 5, 6, 7] and references are therein. These inequalities play a vital role in the study of various branches of mathematics.

In the paper [14], Quac Anh Ngo and the authors obtained the following type of inequality and its variants.

Theorem 1.1. If f(x) is a non-negative continuous function on [0; 1] and f(x) $\geq$ x for all x$\in$ 2 [0; 1]; then

$$\int_0^1 f^{n+1}(x)dx \geq \int_0^1 x^n f(x)dx, \quad for\ all\ n \in \mathbb{N} \tag{1.1}$$

In this paper, we establish some integral inequalities of the type (1.1). In what follows, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = (0,\infty)$ is the given a subset of $\mathbb{R}$. Throughout this paper, all the functions which appear in the inequalities are assumed to be real-valued.

2. General integral inequalities:

In this section, we state and prove some new integral inequalities.

Lemma 2.1. If f(x) is a non-negative continuous function on [0,b], $b \in \mathbb{R}_+$,then

$$f^{n+1}(x) + nx^{n+1} \geq (n+1)x^n f(x); \text{ for all } n \in \mathbb{N} : \tag{2.1}$$

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Proof. By using relation between arithmetic and geometric means, we obtain

$$f^{n+1}(x) + nx|^{n+1} = \frac{(n+1)(f^{n+1} + x^{n+1} + x^{n+1} + \ldots.. + x^{n+1})}{n+1}$$
$$\geq (n+1)(f^{n+1}x^{n+1}x^{n+1}x^{n+1}\ldots.x^{n+1})^{\frac{1}{n+1}}$$
$$= (n+1)(n+1)x^n f(x)$$

which yields (2.1). This completes the proof.

Lemma 2.2. Let f(x) be a non-negative continuous function on [0; b] $b \in \mathbb{R}_+$ Then

$$i) \text{ if } f(x) \geq x \text{ for all } x \in [0,b], \text{then } \int_0^b x^{n+1} f(|x)dx \geq \frac{b^{n+3}}{n+3}, \quad \text{for all } n \in \mathbb{N}$$

$$ii) \text{ if } f(x) \leq x \text{ for all } x \in [0,b], \text{then } \int_0^b x^{n+1} f(|x)dx \leq \frac{b^{n+3}}{n+3}, \quad \text{for all } n \in \mathbb{N}$$

Proof. (i) We have

$$\int_0^b x^n \int_x^a f(t)dt \ dx = \frac{1}{n+1}\int_0^b \int_x^a f(t)dt \ d(x^{n+1}) = \frac{1}{n+1}\int_0^b x^{n+1} f(x)dx \tag{2.2}$$

Which yields

$$\int_0^b x^{n+1} f(x)dx = (n+1)\int_0^b x^n \int_x^a f(t)dt \ dx \tag{2.3}$$

On the other hand using condition f(x) $\geq$x, we obtain

$$\int_0^b x|^n \int_x^b f(t)dt \ dx \geq \int_0^b x^n \frac{b^2 - x^2}{2} dx = \frac{b^{n+3}}{(n+1)(n+3)} \tag{2.4}$$

Therefore from (2.3) and (2.4), yields

$$\int_0^b x^{n+1} f(x)dx \geq \frac{b^{n+3}}{n+3} \tag{2.5}$$

(ii) From the condition f(x) $\leq$ x; we obtain

$$\int_0^b x^n \int_x^a f(t)dt \ dx \leq \int_0^b x^n \frac{b^2-x^2}{2} dx = \frac{b^{n+3}}{(n+1)(n+3)} \tag{2.6}$$

Therefore from (2.3) and (2.6), yields

$$\int_0^b x^{n+1} f(x)dx \geq \frac{b^{n+3}}{n+3} \tag{2.7}$$

This completes the proof.

Theorem 2.3. If f(x) is a non-negative continuous function on [0; b] and f(x) $\geq$ x for all x $\in$ [0; b]; then

$$\int_0^b f^{n+1}(x)dx \geq \int_0^b x^n f(x)dx, \quad for\ all\ n \in \mathbb{N} \tag{2.8}$$

Proof. From the Lemma 2.1, we have

$$\int_0^b f^{n+1}(x)dx + n\int_0^b x^{n+1}dx \geq (n+1)\int_0^b x^n f(x)dx, \quad for\ all\ n \in \mathbb{N} \tag{2.9}$$

On the other hand by using the Lemma 2.2, we obtain

$$(n+1)\int_0^b x^n f(x)dx = n\int_0^b x^n f(x)dx + \int_0^b x^n f(x)dx \geq \frac{nb^{n+2}}{n+2} + \int_0^b x^n f(x)dx \tag{2.10}$$

Therefore from (2.9) and (2.10), we get

$$\int_0^b f^{n+1}(x)dx + n\int_0^b x^{n+1}dx \geq \frac{nb^{n+2}}{n+2} + \int_0^b x^n f(x)dx \tag{2.11}$$

which yields

$$\int_0^b f^{n+1}(x)dx + \frac{nb^{n+2}}{n+2} \geq \frac{nb^{n+2}}{n+2} + \int_0^b x^n f(x)dx \tag{2.12}$$

Thus

$$\int_0^b f^{n+1}(x)dx \geq \int_0^b x^n f(x)dx \tag{2.13}$$

This complete the proof.

Remark 2.1. If b = 1, then Theorem 2.3 reduces to the inequality (1.1) which established by Quac Anh Ngo, Du Duc Thang, Tran Tat Dat and Dang Anh Tuan in [14].

Theorem 2.4. If f(x), g(x) are non-negative continuous functions on [0; b] and f(x)g(x) $\geq$ x for all x $\in$ [0 , b], then

$$\int_0^b f^{n+2}(x)dx + \int_0^b g^{n+2}(x)dx + n\int_0^b x^{n+2}dx \geq (n+2)\int_0^b x^n f(x)g(x)dx \quad for\ all\ n \in \mathbb{N}$$

The proof of the Theorem 2.4 is similar to the proof of the Theorem 2.3 therefore, we omit it here.

Theorem 2.5. If f(x) is a non-negative continuous function on [0; b] and f(x) $\geq$ x for all x $\in$ [0, b], then

$$\int_0^b f^{n+1}(x)dx \geq \int_0^b x f^n(x)dx, \quad for\ all\, n \in \mathbb{N} \tag{2.14}$$

Proof. It is known that

$$0 \leq (f^n(x)-x^n)(f(x)-x) = f^{n+1}(x)+x^{n+1}-x^n f(x)-xf^n(x), \ \ for\ all\ \ x \in (0,b),$$

that is

$$f^{n+1}(x)+x^{n+1} \geq x^n f(x)+xf^n(x), \ \ for\ all\ \ x \in (0,a) \tag{2.15}$$

Thus

$$\int_0^b f^{n+1}(x)dx + \int_0^b x^{n+1}dx \geq \int_0^b x^n f(x)dx + \int_0^b xf^n(x)dx, \ \ for\ all\, n \in \mathbb{N} \tag{2.16}$$

By using the Lemma 2.2, we obtain

$$\int_0^b f^{n+1}(x)dx + \int_0^b x^{n+1}dx \geq \frac{b^{n+2}}{n+2} + \int_0^b xf^n(x)dx, \ \ for\ all\, n \in \mathbb{N} \tag{2.17}$$

that is

$$\int_0^b f^{n+1}(x)dx + \frac{b^{n+2}}{n+2} \geq \frac{b^{n+2}}{n+2} + \int_0^b x^n f(x)dx, \tag{2.18}$$

which gives the conclusion. This completes the proof.

Theorem 2.6. If f(x); g(x) are non-negative continuous functions on [0, b] with f(x) + g(x) $\geq$ x for all x $\in$ [0, b], then

$$\int_0^b f^{n+1}(x)dx + \int_0^b g^{n+1}(x)dx + \frac{b^{n+2}}{n+2} \geq \int_0^b x[f^n(x)+g^n(x)]dx, \ \ for\ all\ \ n \in \mathbb{N}$$

The proof of the Theorem 2.6 is similar to the proof of the Theorem 2.5 therefore, we omit it here.

3. More General integral inequalities

In this section, we establish some integral inequalities using Cauchy inequality. We need the following version of the Cauchy inequality.

Lemma 3.1. Let α and β be positive real numbers satisfying $\alpha+\beta=1$. Then for every positive real numbers x and y, we always have

$$\alpha x + \beta y \geq x^\alpha y^\beta \tag{3.1}$$

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Theorem 3.2. If f(x) is a continuous function on [0, b] and f(x) $\geq$x for all x $\in$ [0, b], then

$$\int_0^b f^{\alpha+1}(x)dx \geq \int_0^b x^\alpha f(x)dx, \tag{3.2}$$

for every positive real number α > 0.

Proof. Using the Lemma 3.1, we obtain

$$\frac{1}{\alpha+1} f^{\alpha+1}(x) + \frac{\alpha}{\alpha+1} x^{\alpha+1} \geq x^\alpha f(x), \text{ for every positive real number } \alpha > 0. \tag{3.3}$$

From (3.3), we obtain

$$\frac{1}{\alpha+1}\int_0^b f^{\alpha+1}(x)dx + \frac{\alpha b^{\alpha+2}}{(\alpha+1)(\alpha+2)} \geq \int_0^b x^\alpha f(x)dx, \tag{3.4}$$

which gives

$$\frac{1}{\alpha+1}\int_0^b f^{\alpha+1}(x)dx + \frac{\alpha b^{\alpha+2}}{(\alpha+1)(\alpha+2)} \geq \frac{1}{\alpha+1}\int_0^b x^\alpha f(x)dx + \frac{\alpha}{\alpha+1}\int_0^b x^\alpha f(x)dx. \tag{3.5}$$

Using the same argument as used in the Lemma 2.2 with f(x) $\geq$x, we obtain

$$\int_0^b x^\alpha f(x)dx \geq \frac{b^{\alpha+2}}{\alpha+2}, \tag{3.6}$$

for every positive real number α > 0.

Hence from (3.5) and (3.6) the conclusion. This completes the proof.

Theorem 3.3. If f(x) is a continuous function on [0; b] and $\sqrt{f(x)g(x)} \geq$ x for all x$\in$ [0,b], then

$$\int_0^b f^{\alpha+1}(x)dx + \int_0^b g^{\alpha+1}(x)dx \geq \int_0^b x^\alpha \sqrt{f(x)g(x)}dx, \tag{3.7}$$

For every positive real number $\alpha > 0$

The proof of the Theorem 3.3 is similar to the proof of the Theorem 3.2 therefore, we omit it.

Theorem 3.4. If f(x) is a continuous function on [0; b] and f(x) $\geq$x for all x $\in$[0, b], then

$$\int_0^b f^{\alpha+1}(x)dx \geq \int_0^b x f^\alpha(x)dx, \tag{3.8}$$

For every positive real number α > 0.

The proof of the Theorem 3.4 is similar to the proof of the Theorem 2.5 therefore, we omit it.

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